

Non-dogmatic social discounting

By ANTONY MILLNER*

The long-run social discount rate has an enormous effect on the value of climate mitigation, infrastructure projects, and other long-term public policies. Its value is however highly contested, in part because of normative disagreements about social time preferences. I develop a theory of ‘non-dogmatic’ social planners, who are insecure in their current normative judgments and entertain the possibility that they may change. Although each non-dogmatic planner advocates an idiosyncratic theory of intertemporal social welfare, all such planners agree on the long-run social discount rate. Non-dogmatism thus goes some way towards resolving normative disagreements, especially for long-term public projects.

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‘I take the problem of discounting for projects with payoffs in the far future...to be largely ethical.’ – Kenneth Arrow (1999)

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‘[T]he list of axioms we use as a basis for our ethical theory can never be more than a tentative list, one always open to possible revision.’

– John Harsanyi (1977)

The social discount rate (SDR) converts the future consequences of public projects into present values, and is thus a critical input to public cost-benefit analysis. Small changes in the SDR can have an enormous effect on the estimated value of public projects with long-run consequences such as infrastructure investments, climate change mitigation measures, and nuclear waste management (Arrow et al., 2013). Yet despite almost a century of economic research on intertemporal public decision-making, opinion is still divided on how costs and benefits that occur more than a few decades in the future should be discounted.

SDRs are related to intertemporal marginal rates of substitution, which quantify the social value of consumption changes in the future. Much of the disagreement about SDRs stems from normative disagreements about the social time preferences that determine these marginal rates of substitution (Drupp et al., 2018a; Dasgupta, 2008). That such disagreements occur should not be surprising – specifying social time preferences requires difficult normative judgments about, for example, the appropriate degree of social impatience and aversion to intertemporal consumption inequalities, and there is no silver bullet specification that is immune to criticism (see Greaves (2017) for a discussion of the arguments). Conversely, as MacAskill (2016) observes, ‘for almost any ethical view, there seems to be *something*

to be said in its favour.’¹

This paper develops a theory of ‘non-dogmatic’ social planners’ time preferences, starting from the premise that no single normative theory of intertemporal social welfare is unassailable, and devotees of all theories should thus exhibit a degree of insecurity in their normative judgments.² Although non-dogmatic planners favour a particular theory of intertemporal social welfare, they admit the possibility that they may be persuaded of the virtues of an alternative theory in the future. Non-dogmatic planners anticipate these possible changes, and internalize the preferences of their future selves. The non-dogmatism of such planners thus manifests both through their willingness to entertain the possibility of a change in their views, and through their unwillingness to act as pure dictators with respect to their future selves.³ Since non-dogmatic planners are always insecure, their normative preferences at any future time τ reflect uncertainty about future preferences at times greater than τ . Persistent normative insecurity coupled with internalization of future preferences thus results in a recursively defined sequence of time preferences, in which current planners’ preferences depend on their

¹An alternative tradition in the literature, often termed the ‘positive approach’, identifies SDRs with observed market interest rates, and thus does not directly engage with normative reasoning. Arrow, Dasgupta and Mäler (2003) however remind us that ‘using market observables to infer social welfare can be misleading in imperfect economies. That we may have to be explicit about welfare parameters...in order to estimate marginal rates of substitution in imperfect economies is not an argument for pretending that the economies in question are not imperfect after all.’ Market imperfections are particularly salient for long-run SDRs. Gollier and Hammitt (2014), for example, explain that ‘the positive approach cannot be applied for time horizons exceeding 20 or 30 years, because there are no safe assets traded on markets with such large maturities.’

²Throughout the paper I use ‘planner’ and ‘theory’ roughly interchangeably. A planner’s time preferences are equivalent to a normative theory of intertemporal social welfare. They are thus distinct from consumer preferences that are inferred by revealed preference, but are an intertemporal analogue of the social welfare functions used in, for example, optimal tax theory.

³Non-dogmatic planners still rank consumption streams using their current preferences, but current preferences depend in part on future preferences.

possible future preferences, each of which is in turn recursively defined.

Crucially, non-dogmatic planners still make idiosyncratic judgments about all the contested normative aspects of intertemporal welfare functions (IWFs),⁴ including utility functions and pure time discount factors. Nevertheless, I show that all non-dogmatic IWFs yield the *same* value of the long-run SDR. Thus, adopting a model of intertemporal evaluation in which planners exhibit some insecurity in their normative judgments ends up resolving disagreements about the evaluation of long-run public projects. The intuition for this finding is developed in an example below. It is a consequence of the fact that each non-dogmatic IWF depends in part on other non-dogmatic IWFs in future periods. Disagreements about long-run SDRs wash out when this nested sequence of interdependent valuations is unravelled backwards to the present, since IWFs mix repeatedly over time.

Formally, the model extends and reinterprets existing models of ‘purely altruistic’ intergenerational time preferences (Ray, 1987; Bergstrom, 1999; Saez-Marti and Weibull, 2005; Galperti and Strulovici, 2017). In these models a single representative agent in the current generation internalizes the preferences of future generations, assuming that each future generation does the same, and that preferences are time invariant. Leading philosophers have long drawn an analogy between present generations’ concerns for future generations, and present selves’ concerns for future selves.⁵ Parfit (1984, p. 319), for example, argues that ‘Like future generations, future selves have no vote, so their interests need to be specially protected.’ The present

⁴IWFs are functions that represent planners’ normative preferences over consumption streams.

⁵Analogies between intergenerational and intrapersonal choice have also proved fruitful in economics (Phelps and Pollak, 1968; Laibson, 1997). See Ray, Vellodi and Wang (2018) for a behavioural model in which agents exhibit concern for future selves.

paper formalizes this analogy in a model of social planners whose normative judgments may change over time. Unlike the intergenerational work cited above, the internalization of future preferences occurs intrapersonally in my model.⁶ Hori (2009) has previously observed in a static framework that increased interdependence between heterogeneous agents who internalize others' preferences can cause their values to get 'closer together'. This paper shows that in any *fixed* model of heterogeneous social planners' time preferences internalization leads to *complete* convergence on the most contested quantity in public cost-benefit analysis: the long-run SDR.

A related literature has studied social choice theoretic approaches to aggregating time preferences⁷(Gollier and Zeckhauser, 2005; Heal and Millner, 2014; Millner and Heal, 2018; Chambers and Echenique, 2018; Feng and Ke, 2018), and the aggregation of heterogeneous opinions on SDRs (Weitzman, 2001; Freeman and Groom, 2015).⁸ Unlike this work, this paper does not specify an aggregation rule that is applied unilaterally by an external analyst. Non-dogmatic planners may disagree on all the normative issues that are sources of contention in discussions of social discounting, and privilege their own theory of intertemporal social welfare. Nevertheless, non-dogmatism causes each planner to account to some extent for alternative theories, so that some aggregation occurs internally in each theory. I show that this is enough to generate consensus on the long-run SDR.

⁶An alternative interpretation of the model that retains an interpersonal flavour is possible, see footnote 15.

⁷A utilitarian aggregation approach leads to representative discount rates that are dominated by the preferences of the most patient agent for long maturities (Gollier and Zeckhauser, 2005). This only occurs in a very special case of the model I develop; in general consensus long-run discount rates are determined by a non-trivial mixture of all non-dogmatic planners' IWFs.

⁸The papers that aggregate SDRs directly do not disentangle heterogeneous beliefs about facts (i.e., consumption growth rates) from disagreements about values. This paper, like the social choice literature, focusses on values.

It is important to emphasize that the model I present is normative: I suggest that planners *should* exhibit some insecurity in their normative judgments, propose a method for them to do so, and use a calibrated version of the model to show how observed disagreements on SDRs might change if they did. The model does not claim to describe the observed behaviour of governments, or the recommendations of ‘experts’ on social discounting. Like much normative work, the paper is an exercise in persuasion. It shows that if advocates of alternative theories of intertemporal social welfare admitted some insecurity in their normative judgments, but were unwilling to give them up entirely, a lot of progress could still be made.

I. A motivating example

The essential features of the model can be illustrated in a simple example. Suppose that there are only two plausible normative theories of intertemporal social welfare, and let planner $i \in \{1, 2\}$ be a devotee of theory i . To establish a baseline model, begin by assuming that at time τ planner i ’s normative preferences over infinite annual consumption streams $\mathbf{C}_\tau = (c_\tau, c_{\tau+1}, c_{\tau+2}, \dots)$ can be represented by an IWF V_τ^i of the following familiar form:

$$(1) \quad V_\tau^i = U^i(c_\tau) + \beta_i V_{\tau+1}^i,$$

where $U^i(c)$ is a utility function and $\beta_i \in (0, 1)$ is a pure time discount factor. These IWFs have the following equivalent representation:

$$(2) \quad V_\tau^i = \sum_{s=0}^{\infty} (\beta_i)^s U^i(c_{\tau+s}).$$

Now consider evaluating a marginal public project with a sequence of annual payoffs $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots)$. Standard results (Dasgupta, Sen and Marglin, 1972; Gollier, 2012) show that the project is welfare improving according to planner i if and only if its net present value is positive, where the net present value of $\boldsymbol{\pi}$ is defined as:

$$(3) \quad NPV^i(\boldsymbol{\pi}) = \sum_{s=0}^{\infty} \pi_s e^{-r^i(s)s},$$

and the social discount *factor* at maturity s is given by the marginal rate of substitution between consumption at times $\tau + s$ and τ , denoted MRS_s^i :

$$(4) \quad e^{-r^i(s)s} = MRS_s^i = (\beta_i)^s \frac{(U^i)'(c_{\tau+s})}{(U^i)'(c_{\tau})}.$$

The social discount *rate* at maturity s according to planner i is

$$(5) \quad r^i(s) = -\frac{1}{s} \ln MRS_s^i.$$

This fundamental quantity tells us how planner i converts safe marginal payoffs at maturity s to present values. Since each planner has an idiosyncratic utility function $U^i(c)$ and pure time discount factor β_i , there is no possibility of them generically agreeing on any part of the term structure of SDRs $r^i(s)$.

Equation (5) can be made more intuitive by assuming that utility functions are iso-elastic (i.e., $(U^i)'(c) = c^{-\eta_i}$), writing $\beta_i = e^{-\rho_i}$, and defining compound annual consumption growth rates g_s via $c_{\tau+s} = c_{\tau} e^{g_s s}$. Substi-

tuting these assumptions into (4–5) we find the famous Ramsey rule:

$$(6) \quad r^i(s) = \rho_i + \eta_i g_s.$$

The first term in this expression is planner i 's pure rate of social time preference, and the second term captures her aversion to intertemporal consumption inequalities, which depends on her elasticity of marginal utility η_i . Planners' adopted values for these parameters constitute primitive normative judgments about how society should trade off consumption at different points in time (Gollier and Hammitt, 2014).

Now consider a variation on the time preferences in (1). Suppose that each planner is a little insecure in her normative judgments, and entertains the possibility that she may be persuaded of the alternative theory of intertemporal social welfare in the future. For concreteness, suppose that the probability of the planners' judgments remaining unaltered next year is w , and the probability of them changing is $1 - w$. How should the planners account for their insecurity today? One answer is that they should simply forget about it. This is perfectly coherent, but amounts to a dogmatic imposition of current preferences on future selves, despite the planners' insecurity about their current, possibly transitory, normative judgments. Normative insecurities have no consequences for SDRs in this case.⁹ A second option is for the planners to adjust their 'raw' preferences by aggregating them with the alternative theory of intertemporal social welfare. However, this places

⁹This is the approach often taken in models of time inconsistent preferences. Sophisticated agents in these models anticipate the *actions* of their future selves, and react optimally to them, but do not incorporate future *preferences* into their own rankings of consumption streams – they are dogmatic. See Galperti and Strulovici (2017) for further discussion of the relationship between time consistency and preference internalization.

current and possible future preferences on an equal conceptual footing *today*, even though the planners are currently devotees of only one theory. We are after a model in which current planners can put all their eggs in one basket. A third option – the one I will pursue – is for the planners to use their current preferences to rank consumption streams, but for those preferences to *internalize* the preferences of future selves. In this interpretation equation (1) is thought of as saying that self τ 's IWF is an additive combination of current utility and the IWF of self $\tau + 1$. If the self at τ admits the possibility that the self at $\tau + 1$ may be persuaded of the alternative theory of intertemporal social welfare, a natural analogue of (1) is:

$$(7) \quad V_{\tau}^1 = U^1(c_{\tau}) + \beta_1(wV_{\tau+1}^1 + (1-w)V_{\tau+1}^2),$$

$$(8) \quad V_{\tau}^2 = U^2(c_{\tau}) + \beta_2((1-w)V_{\tau+1}^1 + wV_{\tau+1}^2),$$

where $w \in (0, 1)$.

In this model planners' insecurity in their current normative judgments causes them to avoid imposing their current preferences on their future self (they only care about the self one year ahead in this example). Current planners account for their future self's preferences directly, and do not just dogmatically value future consumption streams using their current preferences, which may be obsolete by the time next year rolls around. Moreover, normative insecurity is persistent: planners' preferences at time $\tau + 1$ themselves reflect uncertainty about preferences at $\tau + 2$, *ad infinitum*. Planners whose IWFs are of the form in (7–8) will be called 'non-dogmatic' – I provide a formal definition below.¹⁰ Note that the IWFs defined by (7–8) still admit

¹⁰The literature on intergenerational altruism uses the terms 'non-paternalistic' or 'pure' to describe agents who internalize others' preferences. I use 'non-dogmatic' both

arbitrary idiosyncratic pure time discount factors and utility functions.

To analyze the coupled system of time preferences in (7–8), define

$$\vec{V}_\tau = \begin{pmatrix} V_\tau^1 \\ V_\tau^2 \end{pmatrix}; \vec{U}_\tau = \begin{pmatrix} U^1(c_\tau) \\ U^2(c_\tau) \end{pmatrix}; \mathbf{F} = \begin{pmatrix} \beta_1 w & \beta_1(1-w) \\ \beta_2(1-w) & \beta_2 w \end{pmatrix}.$$

Then we can write (7–8) as:

$$(9) \quad \vec{V}_\tau = \vec{U}_\tau + \mathbf{F}\vec{V}_{\tau+1} = \sum_{s=0}^{\infty} \mathbf{F}^s \vec{U}_{\tau+s}.$$

Planners' attitudes to consumption changes in the distant future depend on the behaviour of \mathbf{F}^s for large s . Since $w \in (0, 1)$, the matrix \mathbf{F} is strictly positive. The Perron-Frobenius theorem (see Sternberg, 2014) then tells us that there is a 2×2 matrix \mathbf{A} , with elements $a^{ij} > 0$, such that

$$(10) \quad \lim_{s \rightarrow \infty} \frac{\mathbf{F}^s}{\mu^s} = \mathbf{A},$$

where $\mu \in (0, 1)$ is the largest eigenvalue of \mathbf{F} . Thus when s is large both planners' weights on future utilities are proportional to μ^s , where μ is a non-trivial mixture of both planners' discount factors.¹¹

To understand the intuition for this result notice that current planners at τ only care about utilities at future times $\tau+1, \tau+2, \dots$ indirectly through a mixture \mathbf{F} of their possible IWFs at $\tau+1$. Planners at $\tau+1$ in turn only care about utilities at times $\tau+2, \tau+3, \dots$ through a mixture \mathbf{F} of their possible

to distinguish my model of intrapersonal internalization from this literature, and because this term is a better fit to the context of this paper, in which planners contend with many plausible theories of intertemporal welfare.

¹¹In this example $\mu = \frac{w(\beta_1 + \beta_2)}{2} + \sqrt{\frac{w^2(\beta_1 + \beta_2)^2}{4} - \beta_1\beta_2(2w - 1)}$.

IWFs at time $\tau + 2$. Thus we see that current planners' attitudes to utilities at time $\tau + s$ are obtained by iterating their possible IWFs at $\tau + s$ backwards to τ , passing through their IWFs at times $\tau + s - 1, \tau + s - 2, \dots, \tau + 1$. With each step back in this iteration the discount factors associated with different theories of intertemporal welfare are mixed by the matrix \mathbf{F} . As the number of mixing operations grows (i.e., as s increases), planners' discount factors become homogenized. For large s the mixing process converges, and both planners' long-run utility weights are proportional to a common factor μ^s . It is the fact that planners anticipate possible changes in their theories of intertemporal social welfare, and form their current preferences with one eye on their future selves, that delivers this result.

Substituting (10) into (9) we see that according to planner i , the marginal rate of substitution between consumption at τ and consumption at distant future times $\tau + s$ is

$$(11) \quad MRS_s^i = \frac{\mu^s [a^{i1}(U^1)'(c_{\tau+s}) + a^{i2}(U^2)'(c_{\tau+s})]}{(U^i)'(c_\tau)}.$$

With a few assumptions we can simplify this expression further. Denote the long-run growth rate of consumption by g , i.e., $c_{\tau+s} = e^{gs}c_\tau$ for large s . In addition, define the long-run pure rate of social time preference $\rho = -\ln \mu$, and assume again that utility functions are iso-elastic (i.e., $(U^i)'(c) = c^{-\eta_i}$). Since $(U^i)'(c_{\tau+s}) \propto e^{-g\eta_i s}$ for large s , MRS_s^i is dominated by the exponential term with the lowest value of $\eta_i g$. Substituting these assumptions into (11), we see that when s is large,

$$MRS_s^i \propto e^{-(\rho + \min\{\eta_1 g, \eta_2 g\})s} \Rightarrow r^i(s) \rightarrow \rho + \min\{\eta_1 g, \eta_2 g\}.$$

Thus, although the non-dogmatic planners may have arbitrary disagreements about pure time discount factors and elasticities of marginal utility, they both agree on the long-run SDR. I will show below that disagreements may reduce substantially even for medium term maturities.

II. The model

I now extend the results above to an arbitrary number of planners, each of whom may account for the preferences of future selves into the indefinite future. Assume that there are $N > 1$ plausible normative theories of intertemporal social welfare. As before I identify planner i with theory i , and denote i 's IWF at time τ by V_τ^i . The vector of IWFs at time τ is denoted by $\vec{V}_\tau = (V_\tau^1, V_\tau^2, \dots, V_\tau^N)$. We will say that the time preferences defined by the sequence $\{\vec{V}_\tau\}_{\tau \in \mathbb{N}}$ are *non-dogmatic* if for all $i = 1 \dots N, \tau \in \mathbb{N}$,

$$(12) \quad V_\tau^i = U^i(c_\tau) + \sum_{s=1}^{\infty} \sum_{j=1}^N f_s^{ij} V_{\tau+s}^j,$$

where $f_s^{ij} \geq 0$ for all $s \geq 1$, and there exists a $t \geq 1$ such that $f_t^{ij} > 0$ for all $i, j = 1 \dots N$. Lemma 1 in the Appendix shows that (12) defines a unique bounded set of time preferences that are non-decreasing in all utilities if

$$(13) \quad \max_i \left\{ \sum_{s=1}^{\infty} \sum_{j=1}^N f_s^{ij} \right\} < 1.$$

I assume this condition from now on.¹²

The definition in (12) encodes three assumptions. First, planners' time

¹²The condition in (13) is sufficient, but not necessary, for the required properties to hold. A necessary and sufficient condition is provided in the appendix, however as this condition is difficult to check in practice we will work with (13). None of the results depend on this simplification.

preferences are forward looking and time invariant; this captures the persistence of normative insecurity, and implies that preferences do not depend on the history of consumption. Second, current planners internalize the preferences of possible future selves, and assign non-zero weight to each plausible theory when imagining what their future preferences might be. Third, preferences are additively time separable. A set of IWFs satisfies these three assumptions if and only if it is of the form in (12).¹³

The intertemporal weight f_s^{ij} in (12) is the product of two terms: the pure time discount factor of planner i at time τ on the IWF of self $\tau + s$ (denoted by β_s^i), and the weight i places on theory j in year $\tau + s$ (denoted by w_s^{ij}):

$$f_s^{ij} = \beta_s^i w_s^{ij},$$

where $\sum_{j=1}^N w_s^{ij} = 1$. In the normative application we consider it is natural to require some parity between the weights w_s^{ij} of different planners, as in the simple example above.¹⁴ This ensures that the model delivers a set of theories that are ‘equally non-dogmatic’, but since this is not required for the main result I do not insist on it in the definition.

There is nothing in the representation (12) that requires us to think of the weights w_s^{ij} as probabilities – at present these weights are merely parameters of the preference representation.¹⁵ If, however, we do interpret these weights

¹³Forward looking time invariant IWFs that internalize future preferences are of the form $V_\tau^i = F^i(c_\tau, V_{\tau+1}^1, \dots, V_{\tau+1}^N, V_{\tau+2}^1, \dots, V_{\tau+2}^N, \dots)$. Galperti and Strulovici (2017) show that IWFs of this kind are time separable if and only if they are of the form in (12).

¹⁴Equation A.32 in the appendix gives a more sophisticated example of ‘parity’ between planners’ intertemporal weights.

¹⁵With some modification (12) could be interpreted as a positive model of a set of altruistic agents, each of whom cares about everyone else’s total wellbeing in every future period. This would require the arguments of utility functions to be idiosyncratic private consumption variables, rather than aggregate social consumption – in this case each agent would have N consumption discount rates at each maturity. If, however, these agents

as beliefs, it is natural to require that those beliefs be *consistent*. Consistency requires that current planners' beliefs about which theories they may adopt in the future cohere with their future selves' beliefs about their own chances of switching from their preferred theory. Let $\text{Prob}_\tau(i \rightarrow j; s)$ denote the probability that planner i at time τ assigns to a switch to theory j after s years. Beliefs are consistent iff

$$\text{Prob}_\tau(i \rightarrow j; s) = \sum_{k=1}^N \text{Prob}_\tau(i \rightarrow k; t) \text{Prob}_{\tau+t}(k \rightarrow j; s - t),$$

for all $\tau \in \mathbb{N}$, $s \geq 2$, $1 \leq t < s$. Lemma 2 in the Appendix shows that non-dogmatic planners have consistent beliefs iff there exists an $N \times N$ stochastic matrix \mathbf{P} such that

$$(14) \quad w_s^{ij} = (\mathbf{P}^s)_{i,j},$$

for all $s \geq 1$. We use this restriction on the weights w_s^{ij} in Section IV below, but the main results do not require it.

III. Results

As in the example above, V_τ^i in (12) has an equivalent representation in terms of sums of future utilities which may be determined by solving the infinite system of equations (12) (see appendix). We write the solution of this system as

$$(15) \quad V_\tau^i = \sum_{s=0}^{\infty} \sum_{j=1}^N a_s^{ij} U^j(c_{\tau+s}),$$

derived their utility from consumption of a public good, (12) would apply unchanged.

where $a_s^{ij} \geq 0$ for all $i, j = 1 \dots N, s \in \mathbb{N}$. The SDR at maturity s according to planner i is:

$$(16) \quad r^i(s) = -\frac{1}{s} \ln MRS_s^i = -\frac{1}{s} \ln \left(\frac{1}{(U^i)'(c_\tau)} \sum_{j=1}^N a_s^{ij} (U^j)'(c_{\tau+s}) \right).$$

Define the elasticity of planner i 's marginal utility function as

$$(17) \quad \eta^i(c) = -c \frac{(U^i)''(c)}{(U^i)'(c)}.$$

If $\eta^i(c)$ is uniformly larger than $\eta^j(c)$, planner i is more averse to intertemporal consumption inequalities than planner j . I assume that $\eta^i(c) \geq 0$, is bounded for all c , and that $\lim_{c \rightarrow \infty} \eta^i(c) > 0$ and $\lim_{c \rightarrow 0} \eta^i(c) > 0$ for all i (I assume that all limits exist). In addition, define the long-run growth rate of consumption to be

$$g = \lim_{s \rightarrow \infty} \frac{1}{s} \ln \left(\frac{c_{\tau+s}}{c_\tau} \right)$$

and let

$$(18) \quad \hat{\eta} = \begin{cases} \min_i \{ \lim_{c \rightarrow \infty} \eta^i(c) \} & \text{if } g > 0 \\ \max_i \{ \lim_{c \rightarrow 0} \eta^i(c) \} & \text{if } g < 0. \end{cases}$$

Finally, let \mathbf{F}_s be an $N \times N$ matrix with elements $(\mathbf{F}_s)_{i,j} = f_s^{ij}$, let $\mathbf{1}_N$ be

the $N \times N$ identity matrix, and define the $NM \times NM$ matrix

$$(19) \quad \Phi_M = \begin{pmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \dots & \mathbf{F}_{M-1} & \mathbf{F}_M \\ \mathbf{1}_N & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_N & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_N & \mathbf{0} \end{pmatrix}.$$

Let $\mu(M) \in (0, 1)$ be the largest eigenvalue of Φ_M .

With these definitions in place the main result can be stated.

PROPOSITION 1: *All non-dogmatic planners agree on the long-run SDR:*

$$(20) \quad \lim_{s \rightarrow \infty} r^i(s) = \hat{\rho} + \hat{\eta}g, \quad \forall i = 1, \dots, N,$$

where $\hat{\rho} = -\lim_{M \rightarrow \infty} \ln \mu(M)$.

The proof of this proposition shows that the requirement in (12) that each planner place positive weight on all theories in some future period is stronger than is needed for the result.¹⁶ The formula in (20) can also be extended to the case where consumption growth is uncertain (see the appendix). The proposition also provides a practical procedure for approximating $\hat{\rho}$: compute $-\ln \mu(M)$ for increasingly large values of M .¹⁷

Proposition 1 provides a simple characterization of the consensus long-run elasticity of marginal utility $\hat{\eta}$. The consensus long-run pure rate of social time preference $\hat{\rho}$ is, however, a much more complex quantity, which

¹⁶It is sufficient for each planner to place positive weight on some other theory in some future period, in such a way that if we look far enough ahead, all planners' preferences influence each other. Planner i need not place positive weight on theory j directly.

¹⁷The appendix shows that $-\ln \mu(M)$ decreases monotonically to $\hat{\rho}$ as M increases.

depends on the full set of intertemporal weights f_s^{ij} . The appendix provides further discussion of $\hat{\rho}$, including some comparative statics results. We will content ourselves with describing two intuitive properties of $\hat{\rho}$ here.

PROPOSITION 2: 1) $\hat{\rho}$ is decreasing in f_s^{ij} for all i, j, s .

2) Suppose that the intertemporal weights f_s^{ij} are given by

$$f_s^{ij}(\epsilon) = \begin{cases} f_s^{ii} & j = i \\ h_s^{ij}(\epsilon) & j \neq i, \end{cases}$$

where the functions $h_s^{ij}(\epsilon)$ are continuous, $h_s^{ij}(\epsilon) > 0$ for $\epsilon > 0$, and $h_s^{ij}(0) = 0$. Let $\hat{\rho}_i$ be planner i 's idiosyncratic long-run rate of pure time preference when $\epsilon = 0$, and let $\hat{\rho}(\epsilon)$ be the consensus long-run rate of pure time preference when $\epsilon > 0$. Then

$$(21) \quad \lim_{\epsilon \rightarrow 0^+} \hat{\rho}(\epsilon) = \min_i \hat{\rho}_i.$$

The first part of the proposition is intuitive. Any increase in f_s^{ij} increases the weight planner i places on future utilities. Since all planners' IWFs depend on planner i 's IWF, all planners are less impatient if f_s^{ij} increases. Thus the consensus long-run rate of time preference decreases if f_s^{ij} increases. The second part of the proposition shows that if all planners assign arbitrarily small, but positive, weight to alternative theories, their consensus long-run rate of time preference is the *lowest* of all of their 'dogmatic' rates. To understand the intuition for this finding, note that although planner i places arbitrarily small weight on theories that do not coincide with her current theory as $\epsilon \rightarrow 0$, each theory still enters into her current IWF V_τ^i for all $\epsilon > 0$. When $\epsilon = 0$ planner j 's weights on future utilities decline

like $e^{-\hat{\rho}^j s}$ as $s \rightarrow \infty$. Thus the planner with the lowest value of $\hat{\rho}^j$ will place exponentially more weight on distant future utilities than any more impatient planner as $s \rightarrow \infty$ when $\epsilon = 0$. Since the most patient planner's preferences are part of each planner's preferences for $\epsilon > 0$, by continuity the consensus long-run rate of time preference must be given by the most patient planner's dogmatic long-run rate of time preference as $\epsilon \rightarrow 0$.¹⁸

Part 2 of Proposition 2 invokes related findings on the aggregation of opinions on SDRs (Weitzman, 2001; Freeman and Groom, 2015), and on the utilitarian aggregation of time preferences (e.g. Gollier and Zeckhauser, 2005). In each of these cases averaging over a distribution of discount factors leads to a 'certainty equivalent' discount rate, or a representative discount rate, that declines to the lowest rate as the time horizon tends to infinity. Proposition 2 differs from these results as it pertains to the long-run SDR in *each theory*, rather than an external analysts' average across preferences or real discount rates. The proposition also shows that $\hat{\rho}$ is only determined by the most patient planner in a very special case of the model, i.e., when planners are 'minimally' non-dogmatic. In all other cases, $\hat{\rho}$ is a non-trivial mixture of the intertemporal weights of all theories.

IV. Consequences for cost-benefit analysis

While Proposition 1 emphasizes the emergence of a consensus on the long-run SDR, this result implies a more general phenomenon that has

¹⁸More technically, a matrix that determines planners' pure time discount factors, call it $\Phi(\epsilon)$, separates into N independent components at $\epsilon = 0$, each of which has a dominant eigenvalue that corresponds to the long-run pure time discount factor of one of the dogmatic planners. The largest eigenvalue of $\Phi(0)$ is simply the largest of these N dominant eigenvalues. When $\epsilon > 0$, $\Phi(\epsilon)$ is primitive and has only one component. Since eigenvalues are continuous functions of matrix elements, the largest eigenvalue of $\Phi(\epsilon)$ converges to the largest of the N dogmatic long-run discount factors as $\epsilon \rightarrow 0$.

relevance for cost-benefit analysis. As (3) shows, calculations of the net present value of public projects depend on the full term structure of SDRs. Since non-dogmatic planners' SDRs $r^i(s)$ converge completely as maturity $s \rightarrow \infty$, they must also exhibit partial convergence at finite maturities. Non-dogmatism may thus reduce disagreement about project NPVs by acting through the entire term structure of the SDR. In this section, I illustrate the effect of non-dogmatism on cost-benefit analysis of public projects in a calibrated numerical model, and also show how quickly disagreements about SDRs may decline with maturity. The results in this section constitute a normative counterfactual: I have argued that advocates of all theories of intertemporal welfare *should* be non-dogmatic, and this section illustrates what might happen to disagreements if they were.

To enable this analysis I will work with data on economists' opinions on the normatively appropriate IWF, collected by Drupp et al. (2018a,b).¹⁹ Although there is no deep reason why economists' normative judgments should be seen as representative of the distribution of plausible theories, they do arguably have an advantage in understanding the quantitative implications of different recommendations for cost-benefit analysis. Rawls (1971), in his notion of 'reflective equilibrium', argues that this is an essential feature of good normative reasoning. For my purposes these economists' opinions merely provide an interesting distribution of informed views on these matters.

¹⁹I assume that the opinions expressed in the survey data do not already account for non-dogmatism. Drupp et al. (2018a) state explicitly that 'we structure the survey around a well-known framework for inter-temporal welfare evaluations: Time Discounted Utilitarianism', and work with an iso-elastic utility function. Only two respondents objected to the survey's request for a constant pure rate of time preference, and none objected to the request for a constant elasticity of marginal utility (personal communication). Both of these quantities would vary with maturity if the respondents were non-dogmatic.

The Drupp et al. (2018a) survey contains 173 complete responses from scholars who have published papers on social discounting. Each respondent gave an opinion on the appropriate values of the parameters of a discounted utilitarian IWF with iso-elastic utility function. The 5-95% ranges of opinions on the pure rate of social time preference and elasticity of marginal utility were $[0, 3.85\%/yr]$ and $[0.2, 3]$ respectively. To calibrate the model I assume that the intertemporal weights $f_s^{ij} = \beta_s^i w_s^{ij}$ in (12) take the following form:

$$(22) \quad \beta_s^i = \gamma_i(\alpha_i)^s, \quad w_s^{ij} = (\mathbf{P}^s)_{i,j}, \quad \text{where } \mathbf{P}_{i,j} = \begin{cases} x & i = j \\ \frac{1-x}{N-1} & i \neq j \end{cases}.$$

The parameter $x \in [0, 1]$ is the probability that planners stick to their current theory next year.²⁰ Conditional on switching, planners assign an equal probability to all other theories. By (14), planners' beliefs are consistent in this model. The values of γ_i and α_i are calibrated so that when $x \rightarrow 1$ non-dogmatic planners' IWFs are a close approximation to a discounted utilitarian IWF, and consistent with the values for the pure rate of social time preference that survey respondents reported. Utility functions $U^i(c)$ are taken to be iso-elastic, with the elasticity of marginal utility calibrated to respondents' reported values.

One subtlety of the calibration procedure is worth pointing out. I have chosen to present the model with an annual time step as discount rate

²⁰In this model, $w_s^{ii} = x \left(\frac{Nx-1}{N-1} \right)^{s-1} + \frac{1}{N} \left(1 - \left(\frac{Nx-1}{N-1} \right)^{s-1} \right)$ for all i , and $w_s^{ij} = \frac{1-w_s^{ii}}{N-1}$ for $j \neq i$. However, the appendix explains that the numerical results in this section are robust to alternative specifications of the weights w_s^{ij} for $s \geq 2$. Note that these are weights on future IWFs, and not on future utilities; utility weights are determined by the solution of the entire system (12). No matter what IWF planner i adopts in year s , it is discounted using her *current* intertemporal weight $\gamma_i(\alpha_i)^s$.

schedules are usually presented at this temporal resolution in practice. If, however, we change the time step we also have to change the values of the dynamical parameters in the model, i.e., consumption growth rates, rates of pure time preference, and in particular the transition probability matrix \mathbf{P} , to reflect the change of units. This procedure is less straightforward in this model of interdependent preferences than in more familiar dynamic models, but can be accomplished. The appendix provides further details on this point, and a full description of the calibration procedures.

Given this calibration methodology, the distribution of term structures for the SDR can be computed for different values of the parameter x . Figure 1a depicts the results of this exercise, assuming a constant consumption growth rate of 2%/yr. The figure shows that disagreements about the SDR could reduce substantially even at medium term maturities if planners were non-dogmatic. Reductions in disagreement are greatest at longer maturities, but are substantial even for maturities of 30 years. When planners' judgments are highly persistent (i.e., x is close to 1) the range of opinions on SDRs expands, but for any $x < 97.5\%$ disagreement is reduced by more than a factor of three at maturities greater than 50 years. In the appendix I demonstrate that the rapid reduction in disagreement depicted in Figure 1a is largely driven by non-dogmatic planners' elasticities of marginal utility (i.e., the analogue of the consumption growth term ηg_s in (6)), and not by their rates of pure social time preference (i.e., the analogue of ρ in (6), which depends only on the intertemporal weights f_s^{ij}). Disagreements about the consumption growth term in the Ramsey rule are significantly larger than disagreements about the pure rate of time preference, but decay rapidly with maturity if planners are non-dogmatic. Disagreements about the pure rate

of time preference are smaller, but decay much more slowly with maturity. More than 90% of the variation in $r(s)$ is attributable to variation in the pure rate of time preference alone for maturities $s > 60$ years, for all the values of x depicted in Figure 1a.

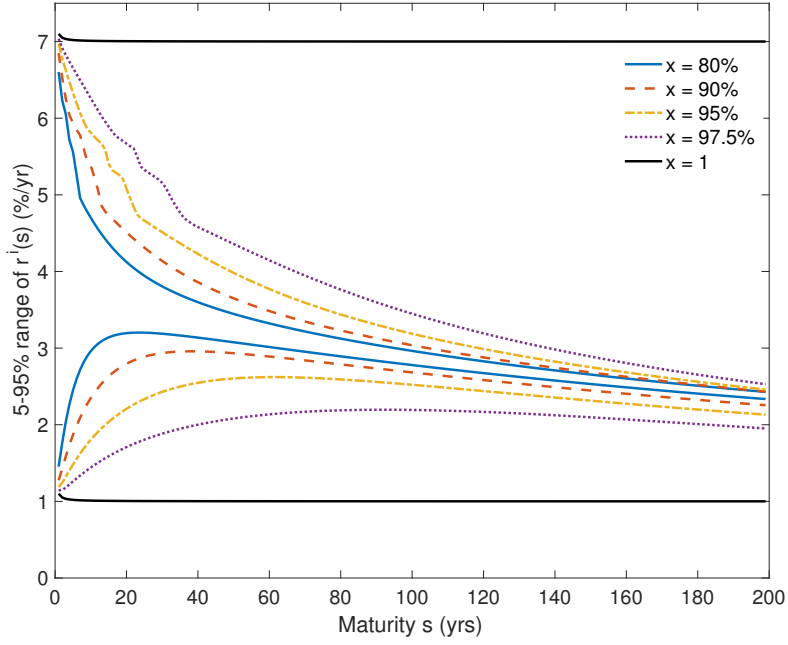
Figure 1b illustrates how non-dogmatism reduces disagreements about the net present values of payoff sequences, as defined by (3). The figure depicts five payoff sequences π .²¹ To quantify the reduction in disagreement about NPVs let $\sigma(\{NPV(\pi; x)\})$ denote the standard deviation of the set of net present values of π according to non-dogmatic planners with weight x in (22), and compute the following ratio for each sequence π :

$$(23) \quad \Gamma(\pi; x) = \frac{\sigma(\{NPV(\pi; x)\})}{\sigma(\{NPV(\pi; 1)\})}.$$

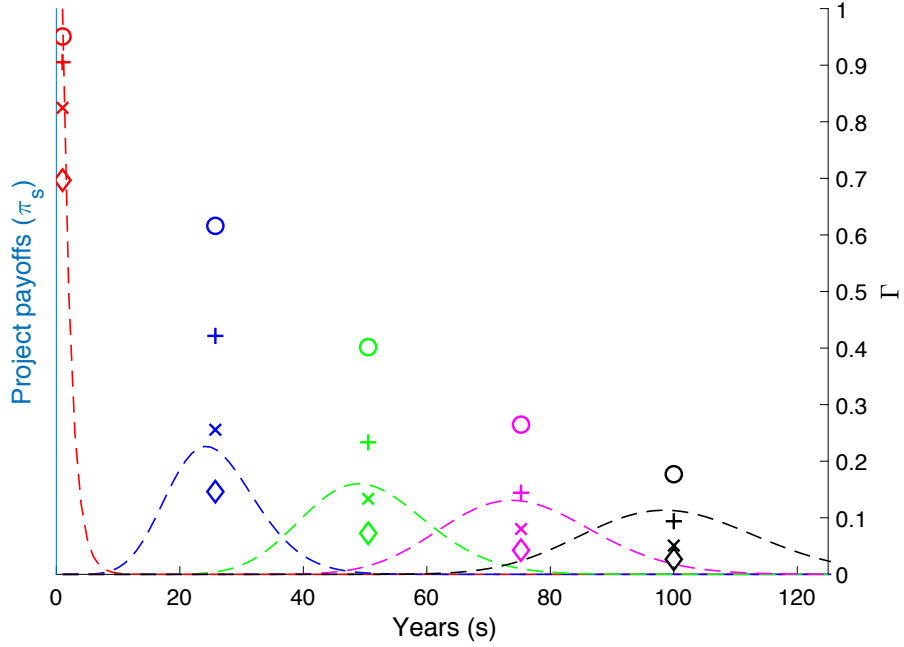
This ratio captures the reduction in disagreement about NPVs, relative to the dogmatic benchmark at $x = 1$. Figure 1b shows that non-dogmatism could substantially reduce disagreements about the value of projects whose payoffs are concentrated at maturities of 30 years or greater, even if $x = 97.5\%$. Reductions in disagreement increase strongly as payoffs move further into the future. For the project on the far right, whose benefits largely occur more than 60 years in future, disagreements are reduced by more than a factor of 5 even if $x = 97.5\%$.

The rate of dissipation of disagreement with maturity depicted in Figure 1 clearly depends on the value of the parameter x . The values $x = 80\%, 90\%, 95\%, 97.5\%$ used in this figure correspond to a change of norma-

²¹These may be thought of as public projects with equal up-front costs but different temporal profiles of benefits. The undiscounted sum of benefits for each project is normalized to 1.



(a) Simulated 5-95% range for non-dogmatic planners' SDRs $r^i(s)$. The solid black curve corresponds to $x = 1$ in (22), dotted purple $x = 97.5\%$, dash-dotted yellow $x = 95\%$, dashed red $x = 90\%$, and solid blue $x = 80\%$. Consumption growth is a constant $2\%/yr$.



(b) Reduction in disagreement about NPVs when planners are non-dogmatic. Each curve in the figure denotes a time sequence of payoffs. The markers centered on each curve denote the values of $\Gamma(\pi, x)$, defined in (23), for this payoff sequence. $\circ, +, \times, \diamond$ denote values of $\Gamma(\pi, x)$ when $x = 97.5\%, 95\%, 90\%, 80\%$ respectively in (22).

Figure 1. : Consequences of non-dogmatism for cost-benefit analysis.

tive views roughly once every 5, 10, 20, and 40 years respectively, on average. Are these plausible values? In order to answer this question we must first recognise that x can be interpreted either as a positive or a normative object. On the one hand, any ethical observer's insecurity in their normative judgments could be construed as a subjective matter; in this interpretation assessing the chance of those views changing is a positive question about that observer's state of mind. There is some suggestive evidence for the occasional changes of heart that are needed to support the paper's conclusions under this interpretation, as professional philosophers' convictions have been shown to correlate with their age (Bourget and Chalmers, 2014).²² On the other hand if we accept the persuasive motivation for the model, i.e., that it is designed to nudge planners into *forming* their normative judgments in a new way, then x plays a normative role. Tweaking this lever shows planners how their normative views on IWFs *should* adjust away from the standard discounted utilitarian framework, given a degree of non-dogmatism that is 'normatively required'. Although both interpretations are consistent with the model, the latter is more in keeping with the ethos of this paper.

Given this, it is reasonable to ask how much non-dogmatism is 'normatively required'. That is something of a meta-ethical question, and readers will doubtless have their own views on it. Requiring planners to admit the *possibility* of a change in their convictions roughly once every 10 or 20 years does not seem like an excessively burdensome prescription (recall that they still have the freedom to discount the preferences of future selves as they see fit). Indeed, the argument that uncertainty or insecurity should play a non-

²²This finding chimes with a witticism often attributed to Georges Clemenceau: 'Not to be a socialist at twenty is proof of want of heart; to be one at thirty is proof of want of head.'

negligible role in normative judgments is not unique to this paper. Catholic theology has grappled with related issues since the 16th century, when the doctrine of ‘probabilism’ was introduced as a guide to action in the face of moral uncertainty (Harty, 1913). Normative uncertainty is currently also a central topic in philosophy, precisely because many have grown weary of old debates that pit ethical theories against one another in a zero-sum game (see e.g. Bostrom, 2009; MacAskill, 2016; MacAskill and Ord, 2018).

V. Conclusion

This paper introduced a normative model of social planners’ time preferences based on a principle of ‘non-dogmatism’. This principle requires advocates of alternative theories of intertemporal social welfare to exhibit a degree of humility when forming their normative judgments: They admit the possibility of a change in their views, and refrain from imposing their current normative judgments on their future selves. The formalism allows advocates of each theory the freedom to express idiosyncratic judgments about all the contested normative aspects of social time preferences. In spite of this, all non-dogmatic theories yield the same value of the long-run social discount rate. As the appropriate value of this quantity has been widely contested and has a powerful influence on the evaluation of public projects with long-run consequences, this analysis may prove useful for policy applications.

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Non-dogmatic Social Discounting

Online Appendix

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A Proof of Lemma 1

Lemma 1. *The system (12) defines a unique bounded set of time preferences, which are non-decreasing in all utilities, if*

$$\max_i \left\{ \sum_{s=1}^{\infty} \sum_{j=1}^N f_s^{ij} < 1 \right\}.$$

Proof. The system of time preferences (12) can be written as a single matrix equation as follows:

$$\begin{pmatrix} V_{\tau}^1 \\ \vdots \\ V_{\tau}^N \\ V_{\tau+1}^1 \\ \vdots \\ V_{\tau+1}^N \\ \vdots \end{pmatrix} = \begin{pmatrix} U^1(c_{\tau}) \\ \vdots \\ U^N(c_{\tau}) \\ U^1(c_{\tau+1}) \\ \vdots \\ U^N(c_{\tau+1}) \\ \vdots \end{pmatrix} + \begin{pmatrix} \vec{0}_N & f_1^{11} & \dots & f_1^{1N} & f_2^{11} & \dots & f_2^{1N} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vec{0}_N & f_1^{N1} & \dots & f_1^{NN} & f_2^{N1} & \dots & f_2^{NN} & \dots \\ \vec{0}_N & \vec{0}_N & f_1^{11} & \dots & f_1^{1N} & f_2^{11} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vec{0}_N & \vec{0}_N & f_1^{N1} & \dots & f_1^{NN} & f_2^{N1} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} V_{\tau}^1 \\ \vdots \\ V_{\tau}^N \\ V_{\tau+1}^1 \\ \vdots \\ V_{\tau+1}^N \\ \vdots \end{pmatrix}$$

where $\vec{0}_N$ is an $1 \times N$ vector of zeros. Letting \vec{X}_{τ} denote the vector on the left hand side of this expression, $\mathbf{\Lambda}$ the infinite dimensional square matrix on the right hand side, and \vec{U}_{τ} denote the vector of U s on the right hand side, we have

$$\begin{aligned} \vec{X}_{\tau} &= \vec{U}_{\tau} + \mathbf{\Lambda} \vec{X}_{\tau} \\ \Rightarrow \vec{X}_{\tau} &= (\mathbf{1}_{\infty} - \mathbf{\Lambda})^{-1} \vec{U}_{\tau}, \end{aligned}$$

where $\mathbf{1}_{\infty}$ is the infinite dimensional identity matrix, and we have assumed that the relevant matrix inverse exists.

In general infinite dimensional matrices do not have unique inverses. However, Lemma 1 in Bergstrom (1999) shows that $\mathbf{1}_{\infty} - \mathbf{\Lambda}$ has a unique bounded inverse with non-negative elements if and only if $\mathbf{1}_{\infty} - \mathbf{\Lambda}$ is a dominant diagonal matrix. A denumerably infinite matrix $\mathbf{1}_{\infty} - \mathbf{\Lambda}$ with $\mathbf{\Lambda} \geq 0$ is said to be dominant diagonal if there exists a bounded diagonal matrix $\mathbf{D} \geq 0$ such that the infimum of the row sums of $(\mathbf{1}_{\infty} - \mathbf{\Lambda})\mathbf{D}$ is positive. Clearly, a sufficient condition for $\mathbf{1}_{\infty} - \mathbf{\Lambda}$ to be dominant diagonal is if $\sum_{s=1}^{\infty} \sum_{j=1}^N f_s^{ij} < 1$ for all i . \square

Although this lemma focusses on providing a sufficient condition that is easy to check, the proof also provides a necessary and sufficient condition: $\mathbf{1}_\infty - \mathbf{\Lambda}$ must be dominant diagonal. This is equivalent to requiring the spectral radius of the linear operator $\mathbf{\Lambda}$ to be less than 1, as this guarantees that the sequence $(\mathbf{1}_\infty - \mathbf{\Lambda})^{-1} = \mathbf{1}_\infty + \mathbf{\Lambda} + \mathbf{\Lambda}^2 + \dots$ converges (Duchin & Steenge, 2009). Checking this condition is however difficult in practice given the infinite dimensionality of $\mathbf{\Lambda}$. I will thus work with the simpler sufficient condition throughout, but the results do not depend on this simplification. The proof of the main proposition in Appendix C only requires the spectral radius of $\mathbf{\Lambda}$ to be bounded above by 1.

B Proof of Lemma 2

We wish to prove that non-dogmatic planners' with preferences (12) have consistent beliefs iff the intratemporal weights w_s^{ij} satisfy (14). In the notation established in the text,

Lemma 2.

$$Prob_\tau(i \rightarrow j; s) = \sum_{k=1}^N Prob_\tau(i \rightarrow k; t) Prob_{\tau+t}(k \rightarrow j; s-t) \quad (\text{A.1})$$

for all $\tau \in \mathbb{N}, s \geq 2, 1 \leq t < s$ if and only if there exists an $N \times N$ stochastic matrix \mathbf{P} such that

$$w_s^{ij} = (\mathbf{P}^s)_{i,j}.$$

Let the beliefs of planners at time τ about the probability of a future self who subscribes to theory i at time $\tau + s - 1$ switching to theory j at time $\tau + s$ be $T_s^{ij,(\tau)}$. Denote the matrix of these transition probabilities by $\mathbf{T}_s^{(\tau)}$. Let $\mathbf{W}_s^{(\tau)}$ be the matrix of time τ planners' beliefs about which theory they will subscribe to at time $\tau + s$, whose i, j element is $Prob_\tau(i \rightarrow j; s)$. Then we have

$$\mathbf{W}_s^{(\tau)} = \mathbf{T}_s^{(\tau)} \mathbf{T}_{s-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)}.$$

Using this relation, (A.1) can be written as the requirement that

$$\mathbf{T}_s^{(\tau)} \mathbf{T}_{s-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)} = \mathbf{T}_{s-t}^{(\tau+t)} \mathbf{T}_{s-t-1}^{(\tau+t)} \dots \mathbf{T}_1^{(\tau+t)} \mathbf{T}_t^{(\tau)} \mathbf{T}_{t-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)}, \quad (\text{A.2})$$

for all τ, t, s . It is clear that a sufficient condition for this to be satisfied is

$$\mathbf{T}_s^{(\tau)} = \mathbf{P}$$

for all τ, s , where \mathbf{P} is an $N \times N$ stochastic matrix. To prove necessity, put $s = 2, t = 1$ in (A.2) to find

$$\mathbf{T}_2^{(\tau)} \mathbf{T}_1^{(\tau)} = \mathbf{T}_1^{(\tau+1)} \mathbf{T}_1^{(\tau)}$$

which implies

$$\mathbf{T}_2^{(\tau)} = \mathbf{T}_1^{(\tau+1)}. \quad (\text{A.3})$$

Putting $s = 3, t = 1$ in (A.2), we find

$$\begin{aligned} \mathbf{T}_3^{(\tau)} \mathbf{T}_2^{(\tau)} \mathbf{T}_1^{(\tau)} &= \mathbf{T}_2^{(\tau+1)} \mathbf{T}_1^{(\tau+1)} \mathbf{T}_1^{(\tau)} \\ \Rightarrow \mathbf{T}_3^{(\tau)} \mathbf{T}_2^{(\tau)} &= \mathbf{T}_2^{(\tau+1)} \mathbf{T}_1^{(\tau+1)}. \end{aligned}$$

and using (A.3) this reduces to

$$\mathbf{T}_3^{(\tau)} = \mathbf{T}_2^{(\tau+1)}.$$

Repeating this process of substitution, we find that a necessary condition for (A.2) to be satisfied is

$$\mathbf{T}_{s+1}^{(\tau)} = \mathbf{T}_s^{(\tau+1)}.$$

Since non-dogmatic planners' preferences are time invariant, it must be the case that

$$\mathbf{T}_s^{(\tau+1)} = \mathbf{T}_s^{(\tau)}.$$

Substituting this relation into the previous equation shows that

$$\mathbf{T}_{s+1}^{(\tau)} = \mathbf{T}_s^{(\tau)}$$

for all τ, s . This implies that the matrix of planners' beliefs $\mathbf{W}_s^{(\tau)}$ must be of the form

$$\mathbf{W}_s^{(\tau)} = (\mathbf{P})^s$$

for all τ .

C Proof of Proposition 1

We prove a more general version of the result in Proposition 1. The proof has two main steps. First we find conditions under which all planners' utility weights a_s^{ij} are proportional to a common discount factor $\hat{\mu}^s$ for large s . We then show that when these conditions are

satisfied all non-dogmatic planners' long-run SDRs are the same.

STEP 1:

Begin by defining the sequence of $N \times N$ matrices

$$\mathbf{F}_s := \begin{pmatrix} f_s^{11} & f_s^{12} & \dots & f_s^{1N} \\ f_s^{21} & f_s^{22} & \dots & f_s^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ f_s^{N1} & f_s^{N2} & \dots & f_s^{NN} \end{pmatrix} \quad (\text{A.4})$$

and the sequences of $N \times 1$ vectors

$$\vec{V}_\tau = \begin{pmatrix} V_\tau^1 \\ V_\tau^2 \\ \vdots \\ V_\tau^N \end{pmatrix}, \quad \vec{U}_\tau = \begin{pmatrix} U^1(c_\tau) \\ U^2(c_\tau) \\ \vdots \\ U^N(c_\tau) \end{pmatrix}. \quad (\text{A.5})$$

Our general model (12) can be written as:

$$\vec{V}_\tau = \vec{U}_\tau + \sum_{s=1}^{\infty} \mathbf{F}_s \vec{V}_{\tau+s}. \quad (\text{A.6})$$

We seek an equivalent representation of this system of the form

$$\vec{V}_\tau := \sum_{s=0}^{\infty} \mathbf{A}_s \vec{U}_{\tau+s}, \quad (\text{A.7})$$

where \mathbf{A}_s is a sequence of $N \times N$ matrices of the form,

$$\mathbf{A}_s := \begin{pmatrix} a_s^{11} & a_s^{12} & \dots & a_s^{1N} \\ a_s^{21} & a_s^{22} & \dots & a_s^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_s^{N1} & a_s^{N2} & \dots & a_s^{NN} \end{pmatrix} \quad (\text{A.8})$$

where a_s^{ij} is the weight planner i at time τ assigns to theory j 's utility function at time $\tau + s$, i.e., $U^j(c_{\tau+s})$.

We now prove the following:

Proposition A.I. *Assume that the condition (13) is satisfied, and that $f_s^{ii} > 0$ for all*

$i = 1 \dots N$, $s = 1 \dots \infty$. Construct a directed graph G with N nodes labelled $1, 2, \dots, N$. Draw an edge from node i to node $j \neq i$ iff $f_s^{ij} > 0$ for at least one $s \geq 1$. If G contains a directed cycle of length N , then there exists a $\hat{\mu} \in (0, 1)$ such that

$$\lim_{s \rightarrow \infty} \frac{a_s^{ij}}{\hat{\mu}^s} = K_{ij} > 0$$

where the K_{ij} are finite constants.

Notice that the definition of non-dogmatic time preferences in (12) automatically implies that the directed cycle condition in this proposition is satisfied (the graph G is complete in this case, i.e., all edges exist). However, the directed cycle condition itself is considerably weaker than is assumed in this definition.

Proof. Substitute (A.7) into (A.6) to find

$$\sum_{s=0}^{\infty} \mathbf{A}_s \vec{U}_{\tau+s} = \vec{U}_{\tau} + \sum_{p=1}^{\infty} \mathbf{F}_p \left(\sum_{q=0}^{\infty} \mathbf{A}_q \vec{U}_{\tau+p+q} \right) \quad (\text{A.9})$$

Equating coefficients of $\vec{U}_{\tau+s}$ in this expression, we see that \mathbf{A}_s must satisfy

$$\mathbf{A}_0 = \mathbf{1}_N \quad (\text{A.10})$$

$$\mathbf{A}_s = \sum_{p=1}^s \mathbf{F}_p \mathbf{A}_{s-p} \text{ for } s > 0. \quad (\text{A.11})$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. The solution of this recurrence relation determines the utility weights a_s^{ij} . It will be convenient to split this matrix recurrence relation into a set of N vector recurrence relations as follows. Let \vec{A}_s^j be the j -th column vector of \mathbf{A}_s , i.e.,

$$\vec{A}_s^j = \begin{pmatrix} a_s^{1j} \\ a_s^{2j} \\ \vdots \\ a_s^{Nj} \end{pmatrix}. \quad (\text{A.12})$$

Define \vec{e}^j to be the unit vector with elements

$$(\vec{e}^j)_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Then (A.11) is equivalent to the N vector recurrence relations

$$\begin{aligned} \vec{A}_0^j &= \vec{e}^j \\ \vec{A}_s^j &= \sum_{p=1}^s \mathbf{F}_p \vec{A}_{s-p}^j \text{ for } s > 0. \end{aligned} \tag{A.13}$$

for $j = 1 \dots N$.

The proof now has the following steps. We consider finite order models, i.e., $\mathbf{F}_{M'} = 0$ for all M' greater than some finite M . We show that if a certain augmented matrix constructed from the matrices $\mathbf{F}_1, \dots, \mathbf{F}_M$ is *primitive*, all planners will have a common long-run pure time discount factor. A square matrix \mathbf{B} is primitive if there exists an integer $k > 0$ such that $\mathbf{B}^k > 0$. We then extend this result to infinite order models by taking an appropriate limit of finite order models. Finally, we show that primitivity of the required matrices in the infinite order case is ensured by the graph theoretic condition in the statement of the proposition.

Begin with the finite order case. Let $M = \max\{s | \exists i, j \ f_s^{ij} > 0\} < \infty$. In this case, for all $s > M$, (A.13) reduces to

$$\vec{A}_s^j = \sum_{p=1}^M \mathbf{F}_p \vec{A}_{s-p}^j. \tag{A.14}$$

Define the $NM \times NM$ matrix

$$\Phi_M = \begin{pmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \dots & \mathbf{F}_{M-1} & \mathbf{F}_M \\ \mathbf{1}_N & 0 & \dots & 0 & 0 \\ 0 & \mathbf{1}_N & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{1}_N & 0 \end{pmatrix} \tag{A.15}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. In addition, define the ‘stacked’ vector

$$\vec{Y}_s^j = \begin{pmatrix} \vec{A}_s^j \\ \vec{A}_{s-1}^j \\ \vdots \\ \vec{A}_{s-M+1}^j \end{pmatrix}$$

Then we can rewrite the M th order recurrence (A.14) as a first order recurrence as follows:

$$\begin{aligned} \vec{Y}_s^j &= \Phi_M \vec{Y}_{s-1}^j \\ \Rightarrow \vec{Y}_{M+s}^j &= (\Phi_M)^s \vec{Y}_M^j. \end{aligned} \tag{A.16}$$

We now assume that Φ_M is a primitive matrix. By the Perron-Frobenius theorem for primitive matrices (Sternberg, 2014), this implies

1. Φ_M has a positive eigenvalue, which we label as $\mu(M)$.
2. All other eigenvalues of Φ_M have complex modulus strictly less than $\mu(M)$.
3. There exists a matrix $\mathbf{C} > 0$ such that

$$\lim_{s \rightarrow \infty} \frac{\Phi_M^s}{[\mu(M)]^s} = \mathbf{C}$$

4. $\mu(M)$ increases when any element of Φ_M increases.
- 5.

$$\mu(M) < \max_i \sum_j \phi_{ij}. \tag{A.17}$$

where ϕ_{ij} is the ij th element of Φ_M .

Since the first N elements of \vec{Y}_s^j coincide with a_s^{ij} , the third of these conclusions implies that

$$\forall i, j, \lim_{s \rightarrow \infty} \frac{a_s^{ij}}{[\mu(M)]^s} = \mathbf{C} \vec{Y}_M^j > 0.$$

To bound the value of $\mu(M)$, note that from point 5 of the Perron-Frobenius theorem in (A.17), and the definition of Φ_M in (A.15), we have

$$\mu(M) < \max_i \left\{ \sum_{s=1}^M \sum_{j=1}^N f_s^{ij} \right\} \tag{A.18}$$

Thus, if

$$\sum_{s=1}^{\infty} \sum_{j=1}^N f_s^{ij} < 1 \quad (\text{A.19})$$

for all i , $\mu(M) < 1$, and hence $\lim_{s \rightarrow \infty} a_s^{ij} = 0$. Thus (13) guarantees that the time preferences (12) are complete (i.e., finite on bounded consumption streams, and hence able to rank arbitrary pairs of bounded consumption streams) for all finite M . This concludes the finite M case.

We now extend this result to the case of infinite M . Assume that there exists an $M' > 0$ such that the matrix Φ_M , defined in (A.15), is primitive for all $M > M'$. For $M > M'$, define

$$\vec{V}_\tau(M) = \vec{U}_\tau + \sum_{s=1}^M \mathbf{F}_s \vec{V}_{\tau+s}(M)$$

and let

$$\hat{\vec{V}}_\tau = \lim_{M \rightarrow \infty} \vec{V}_\tau(M).$$

Define the equivalent representations of these preferences by

$$\vec{V}_\tau(M) = \sum_{s=0}^{\infty} \mathbf{A}_s(M) \vec{U}_{\tau+s} \quad (\text{A.20})$$

$$\hat{\vec{V}}_\tau = \sum_{s=0}^{\infty} \hat{\mathbf{A}}_s \vec{U}_{\tau+s} \quad (\text{A.21})$$

In addition, let $\mu(M)$ be the Perron-Frobenius eigenvalue of Φ_M . We begin by proving that:

Lemma 3.

$$\hat{\mu} := \lim_{M \rightarrow \infty} \mu(M) \text{ exists.} \quad (\text{A.22})$$

Proof. Consider the eigenvalue $\mu(M+1)$, where $M > M'$. This is the Perron-Frobenius eigenvalue of Φ_{M+1} . The M -th order preferences $\vec{V}_\tau(M)$ are equivalent to an $M+1$ th order model, with $\mathbf{F}_{M+1} = 0$. The matrix Φ_M , which controls the asymptotic behavior of $\vec{V}_\tau(M)$ can thus be thought of as an $N \times (M+1)$ matrix, where the last M rows and columns are zeros. Call this matrix $\tilde{\Phi}_{M+1}$. The matrix Φ_{M+1} , associated with the asymptotic behavior of $\vec{V}_\tau(M+1)$, has entries that are strictly larger than those of $\tilde{\Phi}_{M+1}$ in at least some elements. Thus, by point 4 in our statement of the Perron-Frobenius theorem, $\mu(M+1) > \mu(M)$. We also know that $\mu(M) < 1$ for all M . Since the sequence

$\mu(M)$ is increasing and bounded above, the monotone convergence theorem implies that $\hat{\mu}$ exists. \square

We have thus proved that if the matrices Φ_M are primitive for $M > M'$,

$$\lim_{M \rightarrow \infty} \lim_{s \rightarrow \infty} \frac{a_{s+1}^{ij}(M)}{a_s^{ij}(M)} = \lim_{M \rightarrow \infty} \mu(M) = \hat{\mu}. \quad (\text{A.23})$$

Note that since (A.17) and (A.19) are strict inequalities, $\hat{\mu} < 1$. We now wish to know whether it is also true that:

$$\lim_{s \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{a_{s+1}^{ij}(M)}{a_s^{ij}(M)} = \hat{\mu}. \quad (\text{A.24})$$

That is, can we change the order of the limits in (A.23)? For limit operations to be interchangeable we require the sequence of functions they operate on to be uniformly convergent. The functions in question here are $V_\tau^i(M)$ and \hat{V}_τ^i , which we can think of as linear functions from the infinite dimensional space $\mathbb{R}^\infty \times \mathbb{R}^N = \{(\vec{U}_\tau, \vec{U}_{\tau+1}, \vec{U}_{\tau+2}, \dots)\}$ to \mathbb{R} . If the sequence of functions $V_\tau^i(M)$ converges uniformly to \hat{V}_τ^i on any bounded subset of $\mathbb{R}^\infty \times \mathbb{R}^N$, then (A.24) will be satisfied. We now prove a second lemma:

Lemma 4. *Let B be a compact subset of $\mathbb{R}^\infty \times \mathbb{R}^N$, and assume that (13) is satisfied. Then $V_\tau^i(M)$ converges uniformly to \hat{V}_τ^i on B .*

Proof. Equation (A.13) shows that for all $s \leq M$, $a_{\tau+s}^{ij}(M) = \hat{a}_{\tau+s}^{ij}$. Let $\bar{U} = \max_j \{\sup_s \{U^j(c_{\tau+s})\}\}$ be the largest component of any $\vec{U} \in B$. For any $\vec{U} \in B$,

$$\begin{aligned} \sup_{\vec{U} \in B} \left| V_\tau^i(M) - \hat{V}_\tau^i \right| &= \sup_{\vec{U} \in B} \left| \sum_{s=1}^{\infty} \sum_{j=1}^N a_{\tau+M+s}^{ij}(M) U^j(c_{\tau+M+s}) - \sum_{s=1}^{\infty} \sum_{j=1}^N \hat{a}_{\tau+M+s}^{ij} U^j(c_{\tau+M+s}) \right| \\ &\leq \sum_{s=1}^{\infty} \sum_{j=1}^N \left[|a_{\tau+M+s}^{ij}(M)| + |\hat{a}_{\tau+M+s}^{ij}| \right] \bar{U} \end{aligned}$$

By Lemma 3, $\hat{\mu} < 1$ also implies $\mu(M) < 1$ for all M , so we know that $\lim_{M \rightarrow \infty} a_{\tau+M+s}^{ij}(M) = 0 = \lim_{M \rightarrow \infty} \hat{a}_{\tau+M+s}^{ij}$ for all i, j . Thus

$$\lim_{M \rightarrow \infty} \sup_{\vec{U} \in B} \left| V_\tau^i(M) - \hat{V}_\tau^i \right| = 0.$$

Hence $V_\tau^i(M)$ converges uniformly to \hat{V}_τ^i . \square

This concludes the infinite order case.

The final step of the proof is to show that if the graph G , defined in the statement of the proposition, has a directed cycle of length N , then there exists an $M' > 0$ such that for all $M > M'$ the matrix Φ_M is primitive. We demonstrate this using a graphical argument.

Consider an arbitrary $R \times R$ matrix B_{ij} , and form a directed graph $H(B)$ on nodes $1 \dots R$, where there is an edge from node i to node j iff $B_{ij} > 0$. The matrix B_{ij} is primitive if there exists an integer $k \geq 1$ such that there is a path of length k from each node i to every other node j in $H(B)$. If $H(B)$ is *strongly connected*, i.e., there exists a path from every node to every other node, then a sufficient condition for B_{ij} to be primitive is for there to be at least one node that is connected to itself.

Now consider our $NM \times NM$ matrices Φ_M . To construct the directed graph $H(\Phi_M)$ associated with Φ_M in a convenient form, follow the following procedure: Construct an $M \times N$ grid of nodes (where N is the number of planners), with node (m, n) representing planner n at time $\tau + m$. For all $m > 1, n$, construct a directed edge from node (m, n) to node $(m - 1, n)$. In addition, construct a directed edge from node $(1, n)$ to node (m', n') if $f_{m'}^{nn'} > 0$.

As an example, take the case $M = N = 3$, i.e., a third order model with three planners. In this case Φ_M is a 9×9 matrix. Assume that $f_s^{ii} > 0$ for all $i, s = 1 \dots 3$, that $f_1^{12}, f_1^{23}, f_1^{31} > 0$, and that $f_s^{ij} = 0$ otherwise. Figure F.1 represents the directed graph associated with the matrix Φ_3 in this case.

Examination of the figure shows that since $f_s^{ii} > 0$, each of the ‘column’ subgraphs $\{(m, 1)\}, \{(m, 2)\}, \{(m, 3)\}, m = 1 \dots 3$ is strongly connected. Moreover, the cycle between columns (the red dashed edges) connects the columns to each other, and causes the entire graph to be strongly connected. Since each node in the first row is connected to itself, the matrix Φ_3 in this example is primitive.

Returning to the general case, suppose that $f_s^{ii} > 0$ for all i and s . From the example in Figure F.1 it is clear that this implies that for each fixed i the subgraph $\{(m, i) | m = 1 \dots \infty\}$ is strongly connected, with each of the nodes $(1, i)$ connected to itself. Thus, if there is a directed cycle between all of the ‘columns’ of the graph $H(\Phi_{M'})$ for some M' , then for all $M > M'$, $H(\Phi_M)$ is strongly connected, and contains nodes that are connected to themselves. Hence for all $M > M'$, Φ_M is a primitive matrix. This concludes the proof. \square

STEP 2:

We now show that when the conditions of Proposition A.I are satisfied, all non-dogmatic theories yield the same long-run SDR, and we compute an explicit formula for this con-

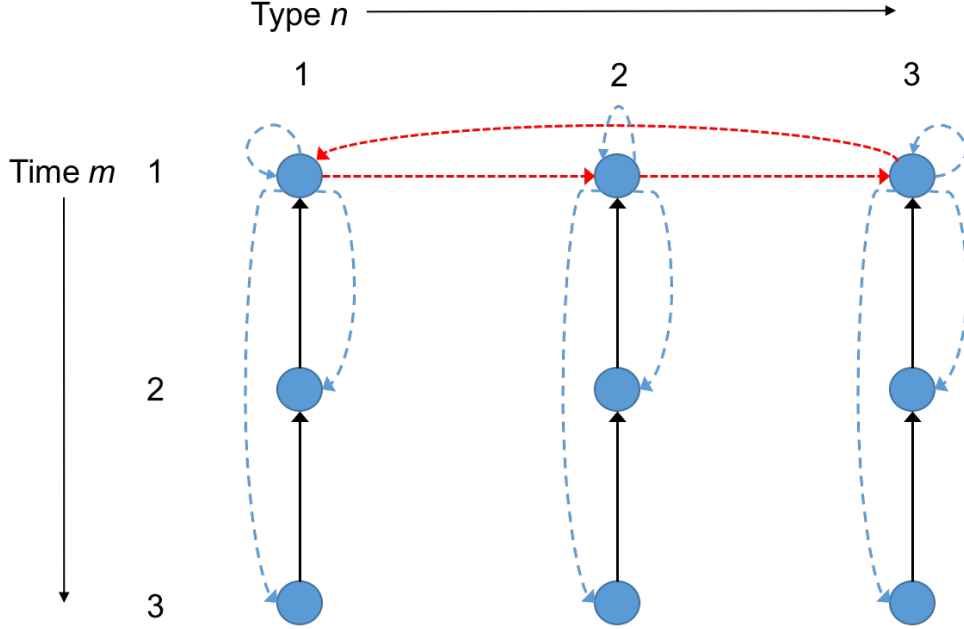


Figure F.1: The directed graph $H(\Phi_3)$ associated with the matrix in our example. The vertical black edges arise from the identity matrices in the definition of Φ_M (see (A.15)). The dashed blue edges arise from $f_s^{ii} > 0$, and the dashed red edges from $f_1^{12}, f_1^{23}, f_1^{31} > 0$.

sensus discount rate.

Begin by defining

$$\hat{\rho} = -\ln \hat{\mu},$$

where $\hat{\mu}$ is defined in (A.22). When the conditions of Proposition A.I hold we know that

$$a_s^{ij} \sim K_{ij}(s)e^{-\hat{\rho}s} \quad (\text{A.25})$$

where \sim denotes asymptotic behaviour as $s \rightarrow \infty$, and the multiplicative factors $K_{ij}(s)$ satisfy $\lim_{s \rightarrow \infty} \frac{1}{s} \ln K_{ij}(s) = 0$.

Now integrate the definition of $\eta^j(c)$ in (17) to find¹

$$(U^j)'(c) = \exp \left(- \int_0^c \frac{\eta^j(x)}{x} dx \right).$$

Make the change of variables $x = c_\tau e^{gs'}$ in the integral in the exponent (recall that g is the

¹In other words, solve the differential equation $-c(U^j)''/(U^j)' = \eta^j(c)$ for $(U^j)'(c)$.

long-run consumption growth rate), and evaluate $(U^j)'(c)$ at $c = c_\tau e^{gs}$ to find

$$(U^j)'(c_\tau e^{gs}) = \exp \left(-g \int_0^s \eta^j(c_\tau e^{gs'}) ds' \right).$$

Defining

$$\hat{\eta}^j = \begin{cases} \lim_{c \rightarrow \infty} \eta^j(c) & g > 0 \\ \lim_{c \rightarrow 0} \eta^j(c) & g < 0 \end{cases} \quad (\text{A.26})$$

we see that the $s \rightarrow \infty$ asymptotic behaviour of marginal utility is given by

$$(U^j)'(c_\tau e^{gs}) \sim L_j(s) e^{-g \hat{\eta}^j s} \quad (\text{A.27})$$

for some functions $L_j(s)$ that satisfy $\lim_{s \rightarrow \infty} \frac{1}{s} \ln L_j(s) = 0$. Combining (A.25) and (A.27), we find

$$\begin{aligned} r^i(s) &= -\frac{1}{s} \ln \left(\frac{1}{(U^i)'(c_\tau)} \sum_{j=1}^N a_s^{ij} (U^j)'(c_{\tau+s}) \right) \\ &\sim -\frac{1}{s} \ln \left(\sum_j K_{ij}(s) L_j(s) e^{-\hat{\rho}s} e^{-\eta^j gs} \right) \\ &\sim \hat{\rho} - \frac{1}{s} \ln \left(\sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs} \right) \end{aligned}$$

Define $\tilde{K}_{ij}(s) = K_{ij}(s) L_j(s)$, and let q be the index of the planner with the lowest (highest) value of $\hat{\eta}^j$ when $g > 0$ ($g < 0$). Then

$$\begin{aligned} \sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs} &= \sum_j \tilde{K}_{ij}(s) e^{-\eta^j gs} \\ &= \tilde{K}_{iq}(s) e^{-\eta^q gs} \left(1 + \sum_{j \neq q} \frac{\tilde{K}_{ij}(s)}{\tilde{K}_{iq}(s)} e^{-(\eta^j - \eta^q)gs} \right) \end{aligned}$$

Since $\eta^j - \eta^q > 0$ for all $j \neq q$ when $g > 0$, and $\eta^j - \eta^q < 0$ for all $j \neq q$ when $g < 0$,

$$\sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs} \sim \tilde{K}_{iq}(s) e^{-\hat{\eta}^q gs},$$

where $\hat{\eta}$ is given by (18). Thus

$$\begin{aligned} r^i(s) &\sim \hat{\rho} - \frac{1}{s} \ln \left(\tilde{K}_{iq}(s) e^{-\hat{\eta}gs} \right) \\ \Rightarrow \lim_{s \rightarrow \infty} r^i(s) &= \hat{\rho} + \hat{\eta}g. \end{aligned}$$

D Consensus long-run SDRs under uncertainty

It is straightforward to extend the proof of Proposition 1 to the case where future consumption is uncertain. If consumption is uncertain non-dogmatic planners' IWFs are simply the expectation over their deterministic IWFs, i.e.,

$$V_\tau^i = \mathbb{E}_{c_{\tau+1}, c_{\tau+2}, \dots} \sum_{s=0}^{\infty} \sum_{j=1}^N a_s^{ij} U^j(c_{\tau+s})$$

where $\mathbb{E}_{c_{\tau+1}, c_{\tau+2}, \dots}$ denotes the expectation over future consumption values, and the coefficients a_s^{ij} are determined by the dynamical system in (A.11), as in the deterministic case.

The analysis of the consensus long-run SDR now proceeds in close analogy to the second part of the proof of Proposition 1. The consensus long-run pure rate of social time preference is unchanged, however examination of the proof shows that we need to account for the effect of expectations on the growth terms in the Ramsey rule.

Under uncertainty planners' marginal rates of substitution between consumption today and consumption s years from now are given by:

$$e^{-r^i(s)s} = MRS_s^i = \frac{\sum_{j=1}^N a_s^{ij} \mathbb{E}_{c_{\tau+s}} (U^j)'(c_{\tau+s})}{(U^i)'(c_\tau)} \quad (\text{A.28})$$

Define a planner specific 'certainty equivalent' long-run growth rate \hat{g}_j by requiring that

$$(U^j)'(e^{\hat{g}_j s} c_\tau) \equiv \mathbb{E}_g (U^j)'(e^{gs} c_\tau) \quad (\text{A.29})$$

as $s \rightarrow \infty$, i.e.,

$$\hat{g}_j \equiv \lim_{s \rightarrow \infty} \frac{1}{s} \log \left[((U^j)')^{-1} \left(\mathbb{E}_g (U^j)'(e^{gs} c_\tau) \right) \right]. \quad (\text{A.30})$$

The long-run consumption growth rate g is uncertain in this expression, and \mathbb{E}_g denotes

expectations over the value of g . In analogy with (A.26), define

$$\hat{\eta}_j(\hat{g}_j) = \begin{cases} \lim_{c \rightarrow \infty} \eta^j(c) & \hat{g}_j > 0 \\ \lim_{c \rightarrow 0} \eta^j(c) & \hat{g}_j < 0. \end{cases}$$

Then for large s , we know from (A.27) that

$$\mathbb{E}_{c_{\tau+s}}(U^j)'(c_{\tau+s}) = (U^j)'(e^{\hat{g}_j s} c_{\tau}) \sim e^{-\hat{g}_j \hat{\eta}_j(\hat{g}_j) s}$$

where \sim denotes $s \rightarrow \infty$ asymptotic behaviour, as before.

As in the deterministic case, we see from (A.28) that planner i 's long-run elasticity of marginal utility is determined by the term that dominates the sum

$$\sum_{j=1}^N a_s^{ij} \mathbb{E}_{c_{\tau+s}}(U^j)'(c_{\tau+s}) \sim \sum_j a_s^{ij} e^{-\hat{g}_j \hat{\eta}_j(\hat{g}_j) s}$$

as $s \rightarrow \infty$. This sum is dominated by the exponential with the minimum value of $\hat{g}_j \hat{\eta}_j(\hat{g}_j)$ (which may be negative), for all i . We thus conclude that the consensus long-run SDR under uncertainty is given by

$$\hat{\rho} + \min_i \{\hat{g}_i \hat{\eta}_i(\hat{g}_i)\} \quad (\text{A.31})$$

As an example of the application of this formula suppose that planners' utility functions are iso-elastic with elasticities of marginal utility η_i , i.e., $(U^i)'(c) = c^{-\eta_i}$. In addition, assume that consumption growth is asymptotically log-normally distributed, i.e.,

$$\log g \sim \mathcal{N}(\mu, \sigma^2).$$

From (A.29) planner i 's certainty equivalent long-run growth rate \hat{g}_i is thus defined by requiring that at large s ,

$$\begin{aligned} e^{-\eta_i \hat{g}_i s} (c_{\tau})^{-\eta_i} &\equiv \mathbb{E}_g e^{-\eta_i g s} (c_{\tau})^{-\eta_i} = e^{-(\eta_i \mu - \frac{1}{2} \eta_i^2 \sigma^2) s} (c_{\tau})^{-\eta_i} \\ \Rightarrow \hat{g}_i &= \mu - \frac{1}{2} \eta_i \sigma^2 \end{aligned}$$

Since elasticities of marginal utility are constant by assumption we know that $\hat{\eta}_i(\hat{g}_i) = \eta_i$, and thus the consensus long-run SDR in this example is given by

$$\hat{\rho} + \min_i \left\{ \mu \eta_i - \frac{1}{2} \eta_i^2 \sigma^2 \right\}.$$

E Proof of Proposition 2

Part 1 of the proposition is immediate from point 4 in our statement of the Perron-Frobenius theorem in Proposition A.I. Part 2 of the proposition follows from the fact that the eigenvalues of a matrix are continuous in its entries. Consider a set of N ‘dogmatic’ models, in which each planner assigns weight only to her own theory in future periods. This set of N independent planners’ time preferences can be represented as a single non-dogmatic set of N planners as in (12), but where $f_s^{ij} = 0$ if $j \neq i$. As in the proof of Proposition A.I, begin by considering a model of finite order M , so that no planner places any weight on any IWF more than M years ahead. Equation (A.16) shows that the asymptotic behaviour of such a model can be described by first order difference equations of the form:

$$\vec{Y}_s^j = \Phi_M^0 \vec{Y}_{s-1}^j.$$

In this case however, the matrix Φ_M^0 , defined in (A.15), is reducible. The largest eigenvalue of Φ_M^0 is the rate of decline of the utility weights of the most patient dogmatic planner in the long-run. As $M \rightarrow \infty$, the set of eigenvalues of Φ_M^0 contains $\hat{\mu}_1^i$, the long-run utility discount factor of planner i , and all eigenvalues of Φ_M^0 are less than or equal to $\max_i \{\hat{\mu}_1^i\}$.

Now consider the continuous set of models with weights $f_s^{ij}(\epsilon)$, where $\epsilon > 0$. Let $\Phi_M(\epsilon)$ be the corresponding Φ_M matrix for this set of models, where by assumption $\lim_{\epsilon \rightarrow 0^+} \Phi_M(\epsilon) = \Phi_M^0$. The consensus long-run discount factor in model ϵ of order M , denoted $\mu_1(\epsilon, M)$ is the largest eigenvalue of $\Phi_M(\epsilon)$. Define

$$\hat{\mu}_1(\epsilon) = \lim_{M \rightarrow \infty} \mu_1(M, \epsilon).$$

We know that this limit exists, due to the proof of Proposition A.I. Since the matrix $\Phi_M(\epsilon)$ is continuous in $\epsilon > 0$, and in the limit as $M \rightarrow \infty$ the largest eigenvalue of $\Phi_M(0) = \Phi_M^0$ is equal to $\max_i \{\hat{\mu}_1^i\}$, we must have

$$\lim_{\epsilon \rightarrow 0^+} \hat{\mu}_1(\epsilon) = \max_i \{\hat{\mu}_1^i\}.$$

Since $\hat{\rho}(\epsilon) = -\ln \hat{\mu}_1(\epsilon)$ by definition, the result follows.

F Comparative statics of the consensus long-run pure rate of social time preference

It is naturally of interest to ask how the consensus long-run pure rate of social time preference $\hat{\rho}$ depends on the intertemporal weights f_s^{ij} . Unfortunately strong comparative statics results on this question are likely out of reach. Technically, we need to understand how the spectral radius (i.e., largest eigenvalue) of the matrices Φ_M from Proposition A.I behaves when we spread out or contract the distribution of weights f_s^{ij} . In order to sign the effect of a spread in the weights we require something akin to a convexity property for the spectral radius. Unfortunately, it is known that the spectral radius of a matrix is a convex function of its diagonal elements, but not of the off-diagonal elements (Friedland, 1981).²

This section describes a special case of the model in which clean comparative statics are possible. Assume that planner i 's intertemporal weights f_s^{ij} depend on a parameter $\lambda_i \in \mathbb{R}^+$, i.e., $f_s^{ij} = f_s^{ij}(\lambda_i)$. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ be the vector of planners' λ parameters, and assume that $\vec{\lambda}$ takes values in a convex subset of \mathbb{R}^{N+} . Using the notation of Proposition A.I we write the matrix of weights f_s^{ij} at a fixed value of s as $\mathbf{F}_s(\vec{\lambda})$, where we now emphasize the dependence of these weights on the parameter vector $\vec{\lambda}$. We will say that preferences are **symmetric in $\vec{\lambda}$** iff for all permutation matrices³ \mathbf{P} ,

$$\mathbf{F}_s(\mathbf{P}\vec{\lambda}) = \mathbf{P}\mathbf{F}_s(\vec{\lambda})\mathbf{P}^T \quad (\text{A.32})$$

for all s , where \mathbf{P}^T is the transpose of \mathbf{P} . Intuitively, if preferences are symmetric in $\vec{\lambda}$, switching any two planners' values of λ is equivalent to switching their entire set of intertemporal weights, as this induces a permutation of the weight matrix $\mathbf{F}_s(\vec{\lambda})$. The parameters λ_i are thus 'sufficient statistics' for planners' intertemporal weights, and switching $\lambda_i \leftrightarrow \lambda_j$ is equivalent to relabelling $i \leftrightarrow j$.

As an example of preferences that are symmetric in $\vec{\lambda}$ consider the following:

$$f_s^{ij} = \begin{cases} \beta(s, \lambda_i)x_s & j = i \\ \beta(s, \lambda_i)\frac{1-x_s}{N-1} & j \neq i \end{cases} \quad (\text{A.33})$$

²Similarly, it is not possible to sign the effect of premultiplying Φ_M by a doubly stochastic matrix, as the spectral radius of a product of two matrices is not sub-multiplicative in general. Gelfand's formula shows that the spectral radius of a matrix product is sub-multiplicative if the matrices in question commute, but this is not much use for our purposes.

³A square matrix is a permutation matrix if each of its rows and each of columns contains exactly one entry of 1, and zeros elsewhere.

where $x_s \in [1/N, 1)$ for all $s = 1 \dots \infty$, and $\sum_{s=1}^{\infty} \beta(s, \lambda) < 1$ for all $\lambda \in I \subset \mathbb{R}^+$. In this model the time dependence of planners' intertemporal weights f_s^{ij} has a common functional form, given by a discount function $\beta(s, \lambda)$ on the IWF of selves s years in the future, where $\lambda > 0$ is a parameter. Variations in planners' attitudes to time are solely due to differences in their values of λ . The parametric model defined in (22), which we used in Section IV of the paper, is of this form if $\gamma_i = \gamma$ for all i .

Let $\hat{\rho}(\vec{\lambda})$ be the consensus long-run pure rate of time preference in a model that is characterized by the parameter vector $\vec{\lambda}$.

Proposition A.II. *Assume that planners' time preferences are symmetric in $\vec{\lambda}$ and that $f_s^{ij}(\lambda)$ is strictly log-convex in $\lambda > 0$ for all i, j, s . Then if the parameter vector $\vec{\lambda}^A$ majorizes⁴ $\vec{\lambda}^B$,*

$$\hat{\rho}(\vec{\lambda}^A) < \hat{\rho}(\vec{\lambda}^B).$$

In words, this result says that if preferences are symmetric in $\vec{\lambda}$, intertemporal weights are log-convex functions of λ , and planners in group A disagree more about the parameter λ than planners in group B, the consensus long-run pure rate of time preference will be *lower* in group A than in group B.

I will provide some interpretation of the log-convexity condition in examples below, but first we turn to the proof.

Proof. The proof relies on the following result due to Kingman (1961): Let $b_{ij}(\theta) \geq 0$ be the elements of a non-negative matrix \mathbf{B} , where $\theta \in \mathbb{R}$ is a parameter. If $b_{ij}(\theta)$ is log-convex in θ for all i, j , the spectral radius of \mathbf{B} is a log-convex function of θ . Remark 1.3 in Nussbaum (1986) observes that Kingman's result can be extended as follows: Let $\vec{\theta}$ be a vector of parameters that takes values in a convex set, and assume that the elements $b_{ij}(\vec{\theta}) \geq 0$ of a matrix \mathbf{B} are log-convex functions of $\vec{\theta}$. Then the spectral radius of \mathbf{B} is log-convex in $\vec{\theta}$.

We will employ the usual trick of working with finite order models first (i.e., setting f_s^{ij} to zero for $s > M$), and taking a limit as $M \rightarrow \infty$ at the end. The consensus long-run pure rate of time preference in a model of order M is determined by the largest eigenvalue of Φ_M , defined in (A.15). Denote this eigenvalue by $\hat{\mu}_M(\vec{\lambda})$. If the matrix elements $f_s^{ij}(\lambda)$ are log-convex functions of the scalar variable λ , then $f_s^{ij}(\vec{\lambda}) = f_s^{ij}(\lambda_i)$ is also a log-convex

⁴ $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$ iff there exists a doubly stochastic matrix \mathbf{H} such that $\vec{\lambda}^B = \mathbf{H}\vec{\lambda}^A$. Intuitively, the elements of $\vec{\lambda}^A$ are 'more spread out' than those of $\vec{\lambda}^B$, and the sums of their elements are equal. See e.g. Marshall (2010) for a discussion of majorization and its relationship to e.g. stochastic orders and inequality measures.

function of the vector of parameters $\vec{\lambda}$. Thus, if $f_s^{ij}(\lambda)$ is log-convex (or identically zero) for all i, j, s , $\hat{\mu}_M(\vec{\lambda})$ is a log-convex function of $\vec{\lambda}$.

The final step of the proof is to observe that because of the symmetry of the set of intertemporal weights in (A.32) the spectral radius must be a symmetric function of $\vec{\lambda}$, i.e., any permutation of the elements of $\vec{\lambda}$ will leave the spectral radius unchanged. This follows since the eigenvalues of a matrix are invariant under the permutations (A.32). Since $\hat{\mu}_M(\vec{\lambda})$ is a log convex, symmetric function of $\vec{\lambda}$, its log is Schur-convex. Since $\hat{\mu}_M(\vec{\lambda}) = e^{-\hat{\rho}_M(\vec{\lambda})}$, this implies that $\hat{\rho}_M(\vec{\lambda})$ is Schur-concave in $\vec{\lambda}$. Thus by the properties of Schur-concave functions, if $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$ we must have

$$\hat{\rho}_M(\vec{\lambda}^A) < \hat{\rho}_M(\vec{\lambda}^B).$$

The final result follows by taking the limit as $M \rightarrow \infty$. □

As an initial example of the application of this result, consider a model in which the discount function $\beta(s, \lambda)$ in the example in (A.33) declines exponentially, i.e.,

$$\beta(s, \lambda) = (1 + \lambda)^{-s}.$$

This discount function satisfies $\log \beta(s, \lambda) = -s \log(1 + \lambda)$, which is strictly convex in λ . Thus the result applies – more disagreement about the parameter λ *decreases* the consensus long-run pure rate of social time preference.

We can extend this finding to a more general class of models by assuming that $\beta(s, \lambda) = \tilde{\beta}(\lambda s)$, i.e., the parameter λ acts to rescale the time variable s . Following Prelec (2004) we will say that $\tilde{\beta}(s)$ exhibits *decreasing impatience* if $\log \tilde{\beta}(s)$ is a convex function of s for $s > 0$. Discount functions that exhibit decreasing impatience have the form $\tilde{\beta}(s) = e^{-h(s)}$ where $h(s)$ is a concave function. The rate of increase of $h(s)$ (which measures impatience) slows as the time horizon s increases.

Corollary 1. *Assume that $\tilde{\beta}(s)$ exhibits decreasing impatience, and that the parameter vector $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$. Then*

$$\hat{\rho}(\vec{\lambda}^A) < \hat{\rho}(\vec{\lambda}^B).$$

Thus, for example, in a hyperbolic model (see e.g. Prelec, 2004) we would have

$$\tilde{\beta}(s) = (1 + s)^{-(1+p)} \Rightarrow \beta(s, \lambda) = \tilde{\beta}(\lambda s) = (1 + \lambda s)^{-(1+p)}$$

where $p > 0$ is a parameter. $\tilde{\beta}(s)$ is log convex in s , so more disagreement about λ reduces the consensus long-run pure rate of time preference in this model.

G Details of calibration

The data I use to calibrate the model and generate the results in Figures 1a and 1b are taken from a recent survey by Drupp et. al. (2018). They surveyed expert economists who have published papers on social discounting, asking for their opinions on, amongst other things, the appropriate values of the pure rate of social time preference and the elasticity of marginal social utility. The distribution of respondents' views on these two parameters is plotted in Figure F.2.

The calibration assumption I use is that the data in Figure F.2 correspond to 'dogmatic' views on the IWF, and in particular that these data correspond to the parameters of a discounted utilitarian IWF with iso-elastic utility function. This assumption is consistent both with the survey authors' description of what they aim to elicit in their survey, and with the participants' responses. See footnote 19 of the main text for further explanation.

The calibration is made slightly delicate by the fact that there is no version of the model in (12) in which planners place non-zero weight on *all* future selves that reduces to a discounted utilitarian IWF. I calibrate the parametric model in (22) so that when the weight on own preferences $x = 1$, planners' time preferences can be represented by a function that is a close approximation to a discounted utilitarian IWF, but still assigns non-zero weight to all future selves.

To calibrate the values of γ_i, α_i in (22), I use the fact that when $x = 1$ the model reduces to a set of N independent intertemporal preferences of the form:

$$V_\tau^i = U^i(c_\tau) + \gamma_i \sum_{s=1}^{\infty} (\alpha_i)^s V_{\tau+s}^i, \quad (\text{A.34})$$

where $\alpha_i \in (0, 1)$ and $\gamma_i \in (0, \frac{1-\alpha_i}{\alpha_i})$. These time preferences have been studied by Saez-Marti & Weibull (2005), and axiomatized by Galperti & Strulovici (2017). It is straightforward to show that they have the following equivalent representation:

$$V_\tau^i = U^i(c_\tau) + \sum_{s=1}^{\infty} \kappa_i^s \left(\frac{1+\gamma_i}{\gamma_i} \right)^{s-1} U^i(c_{\tau+s}), \text{ where } \kappa_i = \alpha_i \gamma_i. \quad (\text{A.35})$$

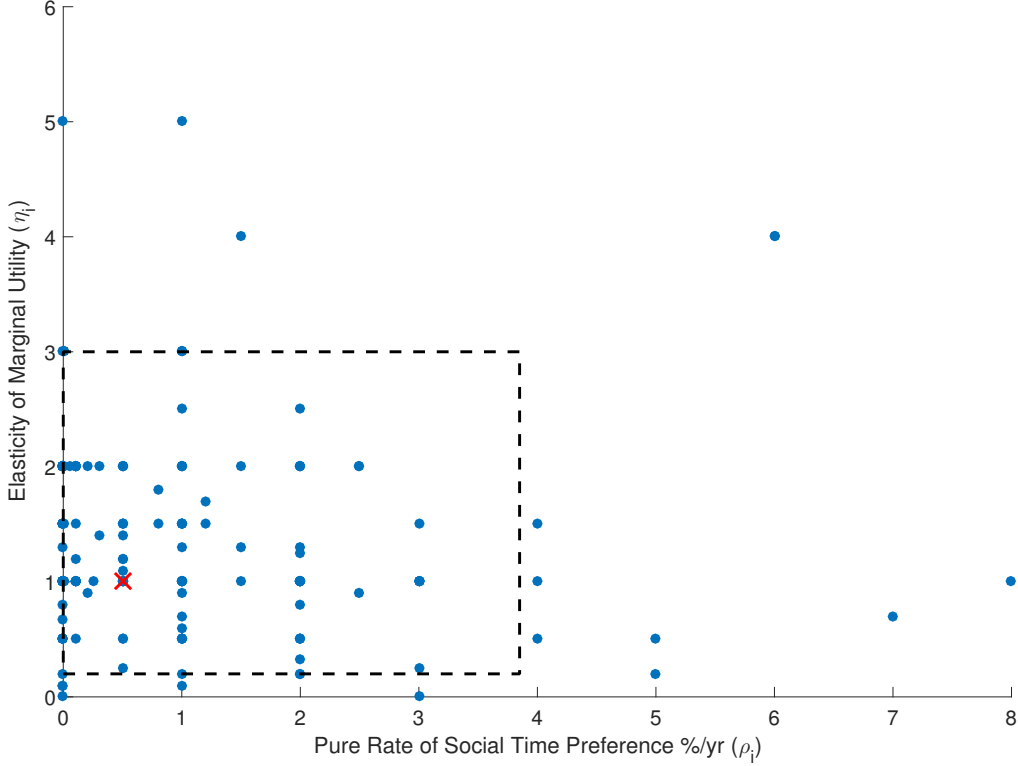


Figure F.2: Experts' recommended values for the pure rate of social time preference (ρ_i), and the elasticity of marginal utility (η_i) for appraisal of long-run public projects, from the Drupp et. al. (2018) survey. 173 responses were recorded. The dashed box depicts data points that fall inside the 5 – 95% ranges of both parameters. The red cross indicates the location of the median values of ρ_i and η_i .

Writing out the sequence of intertemporal utility weights in this model explicitly,

$$1, \kappa_i, \left(\frac{1+\gamma_i}{\gamma_i}\right) \kappa_i^2, \left(\frac{1+\gamma_i}{\gamma_i}\right)^2 \kappa_i^3, \left(\frac{1+\gamma_i}{\gamma_i}\right)^3 \kappa_i^4, \dots, \quad (\text{A.36})$$

it is clear that if we take the limit as $\gamma_i \rightarrow \infty$ of this model holding κ_i fixed, we recover discounted utilitarian time preferences with discount factor κ_i . For any finite γ_i the preferences in (A.35) are quasi-hyperbolic, with a short run pure time discount factor given by κ_i , and a long-run pure time discount factor given by $\left(\frac{1+\gamma_i}{\gamma_i}\right) \kappa_i$.

Recall that the data in Figure F.2 correspond to the parameters of a discounted utilitarian IWF, and that our calibration assumption is that these data correspond to the

$x \rightarrow 1$ limit of the non-dogmatic model (22). The sequence in (A.36) shows that to ensure consistency with the calibration assumption we must calibrate κ_i so that

$$\kappa_i = e^{-\rho_i}, \quad (\text{A.37})$$

where ρ_i is survey respondent i 's recommended value for the pure rate of social time preference. In addition, we must choose γ_i sufficiently large that the model closely approximates discounted utilitarian time preferences. Notice from (A.36) that the discount factor of planner i for $s > 1$ is given by

$$(1 + \gamma_i^{-1})\kappa_i \approx e^{-(\gamma_i^{-1} + \rho_i)}$$

when γ_i^{-1} is small. Thus $\gamma_i^{-1} = 1\%$, for example, corresponds to an additional 1%/yr discount rate on the long-run future, over and above the short run discount rate ρ_i . Thus if γ_i^{-1} is too large, the model will provide a poor fit to a discounted utilitarian IWF when $x = 1$, since non-dogmatic planners will exhibit sharply quasi-hyperbolic time preferences in this case. To ensure that the model is a close approximation to discounted utilitarianism when $x = 1$, but also that all planners place non-zero weight on all future selves' IWFs (which requires γ_i be finite), we must pick γ_i^{-1} to be small but non-zero for all i , i.e., $\gamma_i^{-1} \approx 0.1\%$. The numerical results presented in the paper are robust to heterogeneity in γ_i^{-1} , provided that none of these parameters is too large relative to respondents' pure rates of social time preference. As stated, γ_i^{-1} *must* be small if the calibrated model is to provide a good approximation to discounted utilitarian IWFs at $x = 1$.

In addition, I assume in line with Drupp et. al. (2018) that planners' utility functions are iso-elastic, i.e.,

$$U^i(c) = \frac{c^{1-\eta_i}}{1-\eta_i} \quad (\text{A.38})$$

for some $\eta_i > 0$. This implies that the elasticity of marginal utility is constant and equal to η_i , and I simply calibrate η_i to be each respondent's preferred value of this elasticity.

The requirement that the calibrated model provide a close approximation to discounted utilitarian IWFs in an appropriate 'dogmatic' limit implies that the results depicted in Figure 1a are robust to alternative specifications of the weights w_s^{ij} for $s > 1$. The reason for this is that, as discussed above (and as is evident from (A.36)), in order for the model to closely approximate discounted utilitarian IWFs at $x = 1$, the calibrated values of γ_i must be large, which in turn implies that the values of α_i must be correspondingly small

since $\kappa_i = e^{-\rho_i} = \gamma_i \alpha_i$, where ρ_i is the observed pure time preference rate recommendation of respondent i . Now notice that the models in (22) can be written as

$$V_\tau^i = U^i(c_\tau) + \gamma_i \left[\alpha_i \sum_{j=1}^N w_1^{ij} V_{\tau+1}^j + (\alpha_i)^2 \sum_{j=1}^N w_2^{ij} V_{\tau+2}^j + \mathcal{O}((\alpha_i)^3) \right].$$

Since $(\alpha_i)^s \ll \alpha_i$ for all $s \geq 2$ if $\alpha_i \ll 1$, it does not much matter how the weights w_s^{ij} behave for $s \geq 2$. Even if a weight x is given to current preferences at *every* future maturity, i.e.,

$$w_s^{ij} = \begin{cases} x & i = j \\ \frac{1-x}{1-N} & i \neq j \end{cases} \quad (\text{A.39})$$

for all $s \geq 1$, the results of the simulations hardly change.⁵

H Changing the model's time step

This section of the appendix describes how to transform the parameters of the model used in Figure 1 when the time step is changed.

For the version of the model in question planners' time preferences took the form

$$V_\tau^i = U^i(c_\tau) + \gamma_i \left[\alpha_i \sum_{j=1}^N (\mathbf{P})_{i,j} V_{\tau+1}^j + (\alpha_i)^2 \sum_{j=1}^N (\mathbf{P}^2)_{i,j} V_{\tau+2}^j + \mathcal{O}((\alpha_i)^3) \right]$$

where \mathbf{P} is the annual transition probability matrix defined in (22), which depends on the parameter x , i.e., the chance of a preference change in a year.

If the model's time step is changed from 1 year to $\Delta T > 0$ years the values of all its dynamical parameters must change as well. Consumption growth rates are multiplied by ΔT , and, as in the calibration methodology set out in Section G above, the values of α_i and γ_i must be recalibrated so that:

$$\begin{aligned} \kappa_i &= \alpha_i \gamma_i = e^{-\rho_i \Delta T}, \\ \gamma_i^{-1} &\approx 0.1\% \times \Delta T \end{aligned}$$

Transforming the matrix \mathbf{P} is more complex. To make the version of the model with time

⁵Planners with beliefs (A.39) do not obey the consistency condition (14), but this has no relevance for this discussion.

step ΔT comparable to the original annual model, we need to find a stochastic matrix \mathbf{Q} such that

$$\mathbf{Q} = \mathbf{P}^{\Delta T}. \quad (\text{A.40})$$

When ΔT is not a positive integer (e.g., if $\Delta T = 1/12$ for a monthly time step) such matrix equations may have no solution, or multiple non-negative solutions. However, in our case the structure of the model ensures that there is a natural ‘ ΔT th power’ of \mathbf{P} for any $\Delta T > 0$, and for all interesting values of the parameter x .

Begin by observing that the eigenvalues of \mathbf{P} are 1 (with algebraic multiplicity 1) and $\frac{Nx-1}{N-1}$ (with algebraic multiplicity $N-1$), and are thus positive provided that $x > 1/N$.⁶ Matrices with positive eigenvalues have a unique ‘principal power’ that satisfies the equation (A.40) and itself has positive eigenvalues (see e.g., Horn & Johnson, 2013). It is essential that transforming the time step of the model does not change the signs of the eigenvalues of the model’s transition probability matrix. If this were not the case the qualitative dynamics of preference change would not be preserved under a change of time step. One could, for example, find that planner’s intratemporal weights w_s^{ij} oscillate with maturity s , where no such behaviour existed before.

Since \mathbf{P} is diagonalizable, it can be written as

$$\mathbf{P} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is a matrix whose j th column corresponds to the j th eigenvector of \mathbf{P} , and \mathbf{D} is a diagonal matrix of corresponding eigenvalues, i.e., $(\mathbf{D})_{1,1} = 1$, $(\mathbf{D})_{j,j} = \frac{Nx-1}{N-1}$ for $j \neq 1$. The principal ΔT th power of \mathbf{P} is given by

$$\mathbf{Q} = \mathbf{V}\mathbf{D}^{\Delta T}\mathbf{V}^{-1}.$$

for any $\Delta T > 0$.

Consider the case $\Delta T = 1/12$, corresponding to a model with a monthly time step. It

⁶The case $x < 1/N$ is not plausible.

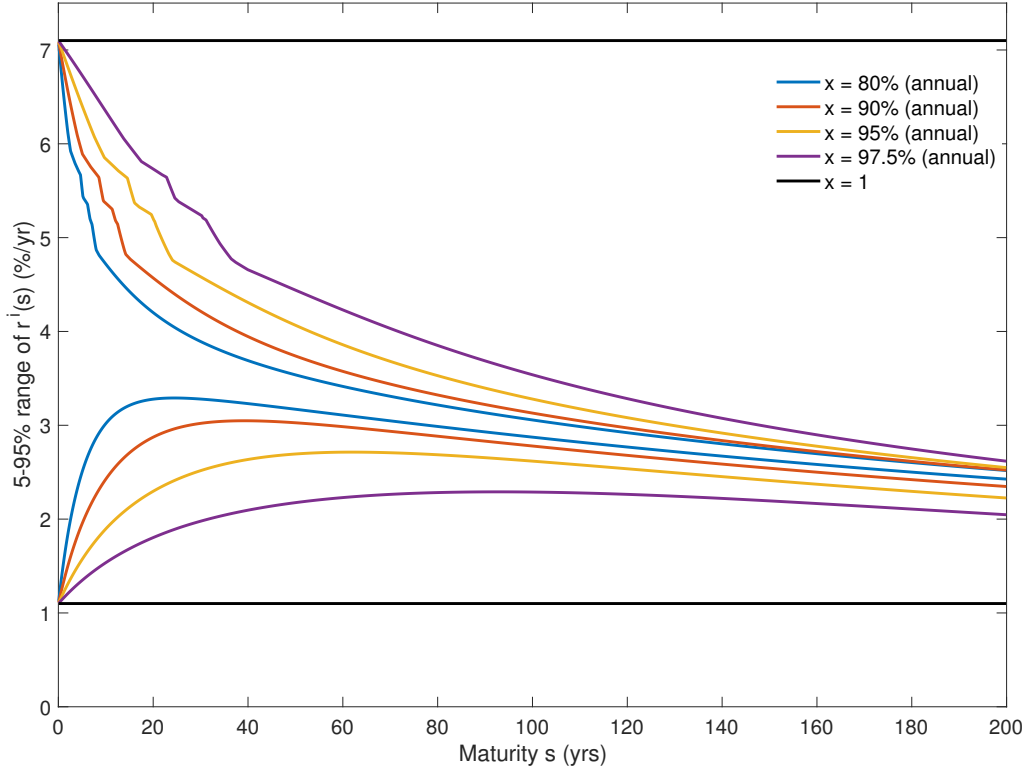


Figure F.3: Replication of Figure 1a in the paper for a monthly time step. To facilitate comparison with Figure 1a monthly discount rates have been converted to annual equivalents (vertical axis), and the horizontal axis is scaled to years, rather than months.

is clear from the definition in the previous equation that raising \mathbf{Q} to the twelfth power yields the original matrix \mathbf{P} , and that \mathbf{Q} has positive eigenvalues. The matrix \mathbf{Q} is the only 12th root of \mathbf{P} that has these properties.⁷

Figure F.3 presents an analogue of Figure 1a in the paper, however this time I have calibrated the model with a monthly time step using the procedure outlined above. The figure shows that there is no appreciable difference between versions of the model defined at different time steps provided that the model parameters are adjusted to reflect the change in time step.

Finally, I note that any version of the model defined with a discrete time step can be

⁷Other solutions of (A.40) have the same basic form as \mathbf{Q} however we may replace any of the entries on the diagonal of $\mathbf{D}^{1/12}$ with any of the twelve complex roots of the corresponding eigenvalue of \mathbf{P} . As there is only one way of choosing these roots so that they are all positive (and real), there is a unique ‘principal power’ of \mathbf{P} .

thought of as an approximation to an underlying continuous model. Preferences could change at any instant, and there is some underlying infinitesimal transition probability matrix that could describe this continuous Markov process. But any discrete approximation of this process, at any temporal resolution, is legitimate – any behaviour of the continuous process, when aggregated up to a discrete time step ΔT by exponentiating the infinitesimal transition matrix, can be replicated by an ‘ab initio’ discrete model with time step ΔT . We lose nothing (at resolution ΔT) in this discrete approximation, although the entries of the discrete transition probability matrix (and hence the weight x) will differ according to the magnitude of ΔT .

I Decomposing non-dogmatic SDRs

This section studies the resolution of disagreement about the two components of the SDR – pure time preference and the consumption growth/inequality aversion term – separately. It shows that much of the rapid convergence of SDRs with maturity shown in Figure 1a is due to exponential convergence in the consumption growth term.

Section C of the appendix showed that the set of IWFs consistent with (12) can be represented by

$$V_\tau^i = \sum_{s=0}^{\infty} \sum_{j=1}^N a_s^{ij} U^j(c_{\tau+s}),$$

where the coefficients a_s^{ij} are determined by the difference equations in (A.11), and $a_0^{ii} = 1, a_0^{ij} = 0$ if $i \neq j$. Planner i ’s SDR at maturity s is

$$r^i(s) = -\frac{1}{s} \ln \left(\frac{\sum_{j=1}^N a_s^{ij} (U^j)'(c_{\tau+s})}{(U^i)'(c_\tau)} \right)$$

We decompose this expression into a pure time preference term and a consumption growth term. Defining

$$\tilde{\rho}^i(s) = -\frac{1}{s} \ln \left(\sum_{j=1}^N a_s^{ij} \right) \tag{A.41}$$

$$G^i(s) = -\frac{1}{s} \ln \left(\frac{\sum_{j=1}^N a_s^{ij} (U^j)'(c_{\tau+s})}{\left(\sum_{j=1}^N a_s^{ij} \right) (U^i)'(c_\tau)} \right). \tag{A.42}$$

we have

$$r^i(s) = \tilde{\rho}^i(s) + G^i(s). \quad (\text{A.43})$$

To understand the meaning of $\tilde{\rho}^i(s)$, notice that $\sum_{j=1}^N a_s^{ij}$ is the total weight on utilities at maturity s in IWF i , i.e., it is a pure time discount factor. Hence $\tilde{\rho}^i(s)$ is IWF i 's pure rate of social time preference at maturity s . To interpret $G^i(s)$ it is helpful to consider the case where the utility functions $U^i(c)$ are iso-elastic as in (A.38). Denoting the compound annual consumption growth rate at maturity s by g_s , we have⁸

$$G^i(s) = -\frac{1}{s} \ln \left(\frac{\sum_{j=1}^N a_s^{ij} e^{-\eta_j g_s s}}{\sum_{j=1}^N a_s^{ij}} \right). \quad (\text{A.44})$$

Consider a hypothetical case in which planners have no normative insecurity, i.e., $a_s^{ij} = 0$ for all $j \neq i$; in this case we see that $G^i(s) = \eta_i g_s$, and we recover the familiar consumption growth term in the Ramsey rule. $G^i(s)$ is the generalization of this term to the non-dogmatic case, i.e., it is the contribution to the discount rate from consumption growth and inequality aversion. Figure F.4 plots the range of values for $\tilde{\rho}(s)$ and $G(s)$ as a function of maturity for the model calibration described in Section G of the appendix. The figure shows two important things. First, disagreements about the consumption growth term are significantly larger, and thus quantitatively more important, than disagreements about the pure rate of social time preference.⁹ Second, although the range of values for $G(0)$ is larger than that for $\tilde{\rho}(0)$, disagreements about this term reduce substantially faster as maturity s increases. The expression for $G^i(s)$ in (A.44) suggests why this occurs. The argument of the log in this expression is a weighted sum of exponential functions, and thus converges exponentially fast to $e^{-\min_j \{\eta_j g_s\} s}$ as s increases. For example, if consumption growth is a constant 2%/yr, and we take $\eta = 2$ as a modal value of η , and $\eta = 0.05$ as the smallest value of η , at a maturity of 50 years we have $e^{-2 \times 0.02 \times 50} = 0.13$, and $e^{-0.05 \times 0.02 \times 50} = 0.95$. Thus values of $\eta_i g_s$ that differ substantially from $\min_j \{\eta_j g_s\}$ receive little weight at long maturities, causing the values of $G(s)$ to converge rapidly.

To relate variation in the components $\tilde{\rho}(s)$ and $G(s)$ back to variation in the SDR

⁸For convenience in this calculation we have chosen units so that current consumption $c_\tau = 1$. This is without loss of generality.

⁹The reader may wonder why the ranges for $\tilde{\rho}(s)$ and $G(s)$ depicted in Figure F.4 do not sum to the range for $r(s)$ in Figure 1a. The answer is that the ranges in Figure F.4 are properties of the *marginal* distributions of $\tilde{\rho}(s)$ and $G(s)$, while the range of their sum $r(s)$ depends on the *joint* distribution of these two quantities. Figure F.4 demonstrates how disagreements about these two independently meaningful quantities reduce as a function of maturity.

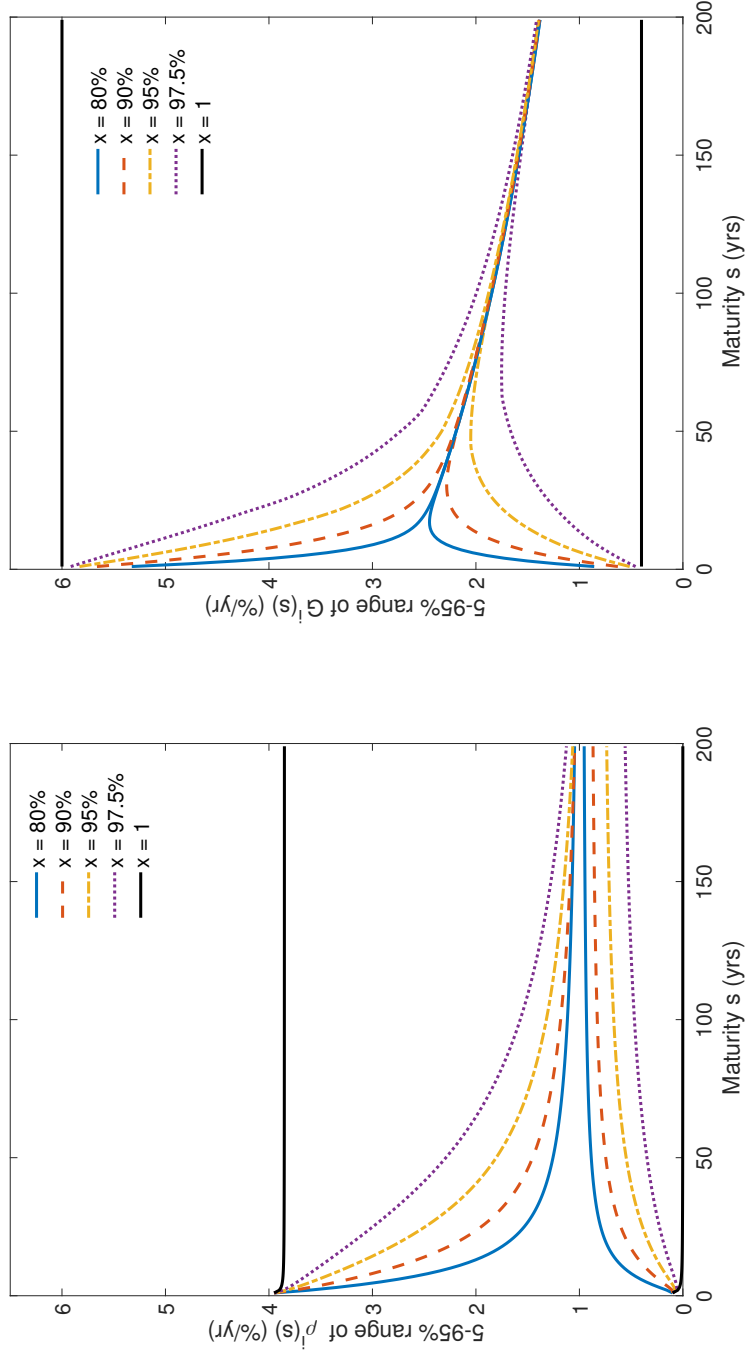
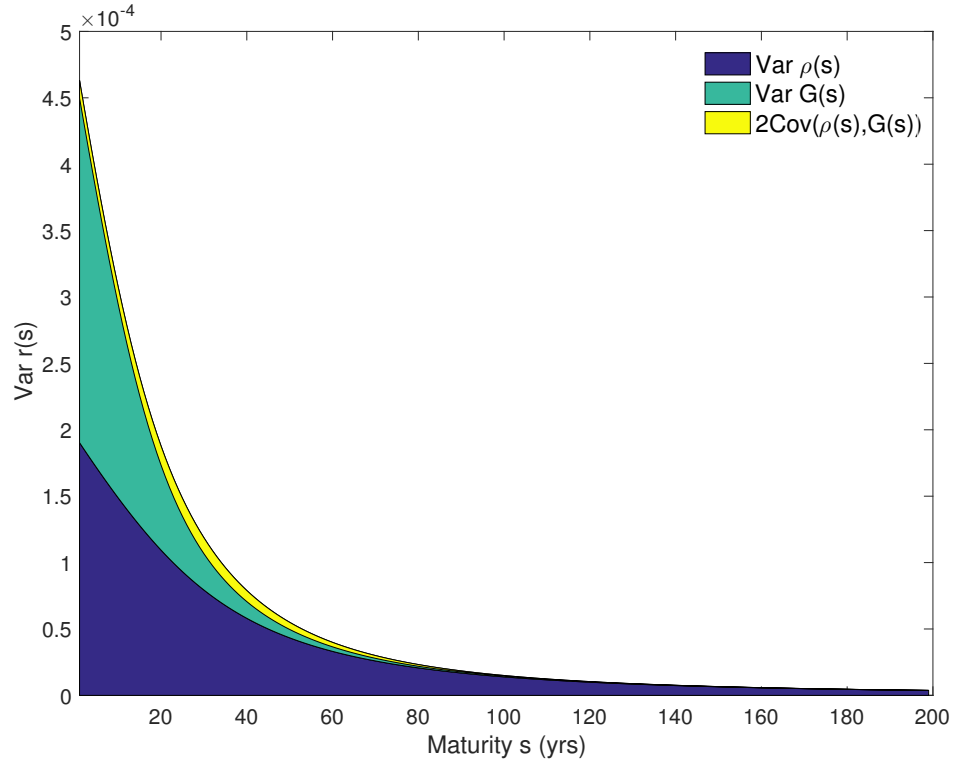


Figure F.4: Range of values for the two components of the SDR – the pure rate of social time preference ($\bar{\rho}(s)$, left), and the consumption growth term ($G(s)$, right) – as a function of maturity s . The model calibration is the same as in Figure 1a.

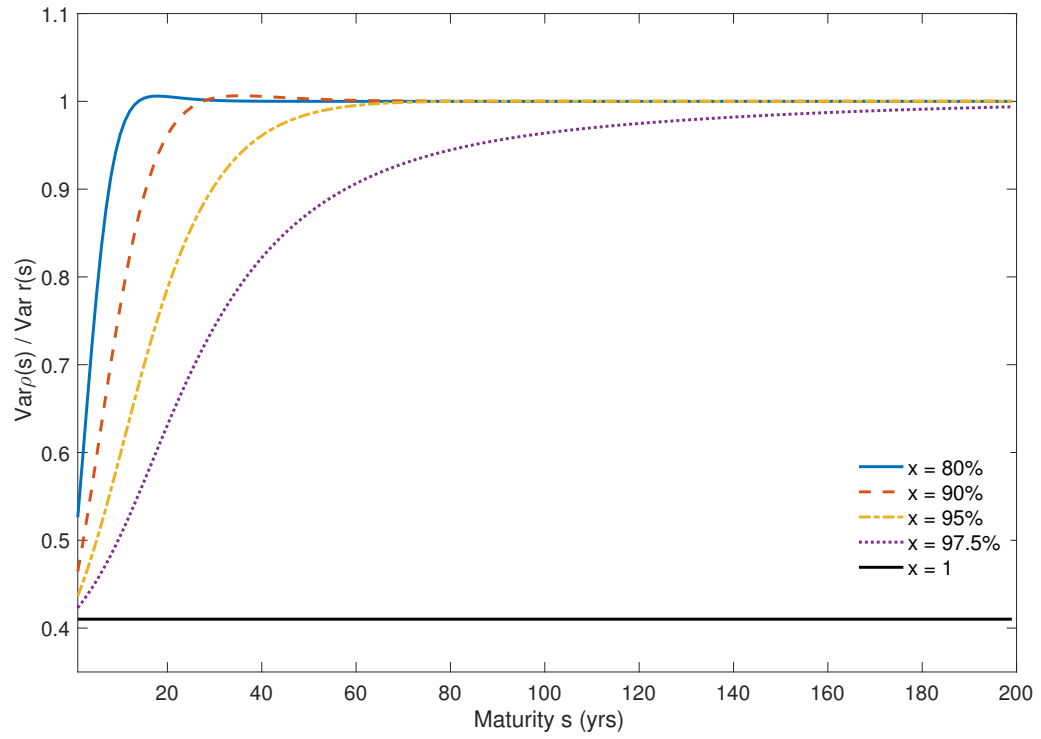
$r(s) = \tilde{\rho}(s) + G(s)$, we make use of the fact that

$$\text{Var } r(s) = \text{Var } \tilde{\rho}(s) + \text{Var } G(s) + 2\text{Cov}\{\tilde{\rho}(s), G(s)\}. \quad (\text{A.45})$$

Figure F.5a breaks the total variance in $r(s)$ into each of these three components at each maturity, for the illustrative case $x = 97.5\%$. This figure confirms that much of the variation in $r(0)$ derives from variation in the growth term $G(0)$, but that as maturities increase disagreements about this term rapidly evaporate. Figure F.5b plots the ratio $\frac{\text{Var } \tilde{\rho}(s)}{\text{Var } r(s)}$ as a function of s for a range of values of x , showing that for all these parameter values almost all the remaining variation in $r(s)$ for $s > 50$ is attributable to variation in $\tilde{\rho}(s)$ – we have almost complete convergence on the dominant $G(s)$ term at these maturities.



(a) Components of the variance of $r(s)$ (see equation (A.45)). $x = 97.5\%$.



(b) Share of the variance of $r(s)$ due to the variance of $\tilde{\rho}(s)$.

Figure F.5: Decomposition of the variance of $r(s)$.

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